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Two remarks on a generalized Davey–Stewartson system

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Abstract

We present two results on a generalized Davey–Stewartson system, both following from the pseudo-conformal invariance of its solutions. In the hyperbolic–elliptic–elliptic case, under some conditions on the physical parameters, we establish a blow-up profile. These conditions turn out to be necessary conditions for the existence of a special "radial" solution. In the elliptic–elliptic–elliptic case, under milder conditions, we show the L^p-norms of the solutions decay to zero algebraically in time for $2 < p < \infty$.

Keywords: Pseudo-conformal invariance; Davey–Stewartson system; Blow-up profile; L^p-stability; Nonlinear Schrödinger equation

1. Introduction

In two space dimensions, the solutions of the Schrödinger equations with cubic nonlinearity (NLS) are invariant under the pseudo-conformal transformation ([4,11] and references therein). In addition to its inherent interests, this invariance has far reaching consequences leading to a better understanding of the blow-up profiles [9,10,12,13]; global existence of the solutions [3]; as well as their L^p-stability [6,13].

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It is well-known that many equations can be expressed as a NLS equation with additional and possibly non-local terms [6,8]. For example, in the elliptic–elliptic and hyperbolic–elliptic cases the Davey–Stewartson (DS) system can be written as

$$iu_t + \delta u_{xx} + u_{yy} = \chi |u|^2 u + bu E(|u|^2)$$

with $\delta = \pm 1$, where E is a linear pseudo-differential operator with a homogeneous symbol of order zero [7]. The non-local term $bu E(|u|^2)$ acts in many ways like a cubic nonlinearity: they have similar scaling properties and the interaction between these two nonlinear terms determine the global behaviour of the solutions [7].

As it is the case for the NLS equation, the solutions of the DS system are invariant under the pseudo-conformal transformation [5,10]. For the elliptic NLS, this invariance plays a key role in understanding the blow-up profile of solutions [9,12,13], whereas in the hyperbolic–elliptic case of DS system an explicit blow-up profile is obtained via the pseudo-conformal invariance [10].

An analogous system has been derived in [2] to model wave propagation in a generalized elastic medium and has been called Generalized Davey–Stewartson (GDS) system. In [1], for the hyperbolic–elliptic–elliptic (HEE) and elliptic–elliptic–elliptic (EEE) cases the GDS system has been expressed as

$$iu_t + \sigma u_{xx} + u_{yy} = \kappa |u|^2 u + \gamma K(|u|^2) u, \tag{1}$$

where in terms of transformed variables,

$$\widehat{K(f)}(\xi) = \alpha(\xi) \hat{f}(\xi)$$

with

$$\alpha = \alpha(\xi_1, \xi_2) = \frac{\lambda \xi_1^4 + (1 + m_1 - 2n)\xi_1^2 \xi_2^2 + m_2 \xi_2^4}{(\lambda \xi_1^2 + m_2 \xi_2^2)(\xi_1^2 + m_1 \xi_2^2)}.$$

The representation given in (1) leads one to expect some of the properties of the NLS equation to remain valid for the GDS system.

The main aim of this note is to highlight the importance of the pseudo-conformal invariance for the GDS system. In the second section, we start by recalling some of the work done for the classification of the GDS system as well as its conserved quantities [1]. Then the solutions are also shown to be invariant under a scale transformation. The corresponding conserved quantity, denoted by $\mathscr{S}$, is used for a derivation of the virial identity. Next, the pseudo-conformal transformation for the GDS system is stated and its invariant, denoted by $\mathscr{M}$, is found. In the third section, we focus on the HEE case for the GDS system. Starting from a solution ansatz in the spirit of [10], a set of conditions are found on the underlying parameters. These conditions also turn out to be necessary conditions for the existence of a "radial" steady-state solution. The pseudo-conformal transformation converts this steady-state solution into a time dependent one which blows-up in finite time. In the last section, following an idea given in [13] we show that for $p > 2$, the L^p-norms of smooth solutions of the GDS system in the EEE case converge to 0 as $t \to \infty$.

2. The GDS system and further invariants

We consider the GDS system as proposed in [2] to model 2 + 1 dimensional wave propagation in a bulk medium composed of elastic material with couple stresses

$$iu_t + \sigma u_{xx} + u_{yy} = \kappa |u|^2 u + \gamma(\varphi_{1,x} + \varphi_{2,y})u, \tag{2}$$

$$\varphi_{1,xx} + m_2\varphi_{1,yy} + n\varphi_{2,xy} = (|u|^2)_x, \tag{3}$$

$$\lambda\varphi_{2,xx} + m_1\varphi_{2,yy} + n\varphi_{1,xy} = (|u|^2)_y, \tag{4}$$

where u is the scaled complex amplitude of the free short transverse wave mode and φ_1 and φ_2 are the scaled free long longitudinal and long transverse wave modes, respectively. The parameters $\sigma, \kappa, \gamma, m_1, m_2, \lambda$ and n are real constants and σ is normalized as $|\sigma| = 1$. The parametric relation $(\lambda-1)(m_2-m_1)=n^2$ follows from the structure of the physical constants and plays a key role in the analysis of these equations. In [1], these equations were classified according to the signs of $(\sigma, m_1, m_2, \lambda)$. Here we will only be considering $(+, +, +, +)$ EEE and $(-, +, +, +)$ HEE cases. An existence and uniqueness result for the initial-value problem with $u_0 \in H^1(\mathbb{R}^2)$ in the EEE case was indicated in [1]. As suggested in [7] this argument can be modified to cover the HEE case as well. Four conserved quantities, corresponding to mass, momentum in the x and y directions and energy, were derived in [1] in the form

$$\begin{aligned}
\mathscr{N} &= \int_{\mathbb{R}^2} |u|^2 \,\mathrm{d}x\,\mathrm{d}y \\
\mathscr{P}_x &= \int_{\mathbb{R}^2} i(u^* u_x - u u_x^*) \,\mathrm{d}x\,\mathrm{d}y \\
\mathscr{P}_y &= \int_{\mathbb{R}^2} i(u^* u_y - u u_y^*) \,\mathrm{d}x\,\mathrm{d}y \\
\mathscr{H} &= \int_{\mathbb{R}^2} \Big\{ \sigma|u_x|^2 + |u_y|^2 + \frac{\kappa}{2}|u|^4 + \frac{\gamma}{2}[(\varphi_{1,x})^2 + m_2(\varphi_{1,y})^2 \\
&\qquad + \lambda(\varphi_{2,x})^2 + m_1(\varphi_{2,y})^2 + n(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})] \Big\} \,\mathrm{d}x\,\mathrm{d}y,
\end{aligned} \tag{5}$$

respectively. The function spaces that the solution $(u, \varphi_1, \varphi_2)$ of the initial-value problem resides in make these conserved quantities mathematically meaningful as well. Let us remark in passing that the same conclusion is much more difficult to draw in the cases where Eqs. (3) and (4) are hyperbolic. In [1] a virial type identity was derived for the solutions of the initial value problem that live in weighted spaces. Using this identity a blow-up result was established for the EEE case.

In addition to the four conserved quantities given in (5) we now show two further invariants. We start by showing that the GDS system admits a scale invariance. Namely, the transformation $(x, y, t, u, \varphi_1, \varphi_2) \to (x', y', t', u', \varphi_1', \varphi_2')$ where

$$\begin{aligned}
&u'(x', y', t') = \mu u(x, y, t), \quad \varphi_1'(x', y', t') = \mu\varphi_1(x, y, t), \\
&\varphi_2'(x', y', t') = \mu\varphi_2(x, y, t) \quad x' = x/\mu, \quad y' = y/\mu, \quad t' = t/\mu^2
\end{aligned} \tag{6}$$

leaves the GDS system invariant. Following Noether's theorem, the corresponding scale invariant is obtained as

$$\mathscr{S} = i \int_{\mathbb{R}^2} \{x(uu_x^* - u^* u_x) + y(uu_y^* - u^* u_y)\}\, \mathrm{d}x\, \mathrm{d}y - 4t\mathscr{H}. \tag{7}$$

When $\sigma = 1$ in (2), the variance, or the moment of inertia, of a solution u is given by

$$\mathscr{V}(t) = \int_{\mathbb{R}^2} (x^2 + y^2)|u|^2\, \mathrm{d}x\, \mathrm{d}y$$

and satisfies

$$\frac{\mathrm{d}\mathscr{V}}{\mathrm{d}t} = 2i \int_{\mathbb{R}^2} \{x(uu_x^* - u^* u_x) + y(uu_y^* - u^* u_y)\}\, \mathrm{d}x\, \mathrm{d}y.$$

Differentiating this equation with respect to t and using (7) results in the virial identity $\mathrm{d}^2\mathscr{V}/\mathrm{d}t^2 = 8\mathscr{H}$. In [1], this identity was derived directly from Eqs. (2)–(4) and was instrumental in establishing the blow-up of the solutions in the EEE case.

Furthermore, the solutions of the GDS system are also invariant under the pseudo-conformal transformation given as $(x, y, t, u, \varphi_1, \varphi_2) \to (x', y', t', u', \varphi_1', \varphi_2')$ where

$$u'(x', y', t') = (a + bt) \exp\left(-ib\, \frac{\sigma x^2 + y^2}{4(a + bt)}\right) u(x, y, t),$$

$$\varphi_1'(x', y', t') = (a + bt)\varphi_1(x, y, t), \quad \varphi_2'(x', y', t') = (a + bt)\varphi_2(x, y, t),$$

$$x' = \frac{x}{(a + bt)}, \quad y' = \frac{y}{(a + bt)}, \quad t' = \frac{c + dt}{a + bt}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}). \tag{8}$$

The conserved quantity corresponding to the pseudo-conformal transformation is obtained by writing the energy in the transformed coordinates

$$\mathscr{H}' = \int_{\mathbb{R}^2} \Big\{\sigma|u'_{x'}|^2 + |u'_{y'}|^2 + \frac{\kappa}{2}|u'|^4 + \frac{\gamma}{2}[(\varphi'_{1,x'})^2 + m_2(\varphi'_{1,y'})^2 + \lambda(\varphi'_{2,x'})^2 + m_1(\varphi'_{2,y'})^2 + n(\varphi'_{1,y'}\varphi'_{2,x'} + \varphi'_{1,x'}\varphi'_{2,y'})]\Big\}\, \mathrm{d}x'\, \mathrm{d}y'.$$

When $a = d = 0$, $b = -1$ and $c = 1$, the energy $\mathscr{H}'$ takes the form

$$\mathscr{M} \equiv 4\mathscr{H}' = \int_{\mathbb{R}^2} \{\sigma|xu + 2i\sigma t u_x|^2 + |yu + 2itu_y|^2 + 2t^2(\kappa|u|^4 + \gamma[(\varphi_{1,x})^2 + m_2(\varphi_{1,y})^2 + \lambda(\varphi_{2,x})^2 + m_1(\varphi_{2,y})^2 + n(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})])\}\, \mathrm{d}x\, \mathrm{d}y$$

and this quantity is conserved with respect to time.

Remark. The conserved quantity $\mathscr{M}$ has been independently derived using Noether's theorem.

3. A blow-up profile in the HEE case

We consider the HEE case in this section and assume that $\sigma = -1$. We seek solutions of the form

$$u(x, y, t) = \frac{1}{1 + Ax^2 + By^2}, \quad \varphi_1(x, y, t) = C_1 \frac{2Ax}{1 + Ax^2 + By^2},$$
$$\varphi_2(x, y, t) = C_2 \frac{2By}{1 + Ax^2 + By^2}, \tag{9}$$

where A, B, C_1, C_2 are arbitrary real constants (see [10] for the case of the DS system). Substituting these into (2)–(4), the following constraints on the physical parameters $\gamma, m_1, m_2, \lambda$ and n and the arbitrary constants are obtained:

$$\lambda = m_1(1 - n), \quad m_2 = 1 - \frac{n}{m_1}, \quad \kappa = -\frac{\gamma}{2}\left(1 + \frac{1}{m_1}\right), \tag{10}$$

$$C_1 = m_1 C_2, \quad A = B = \frac{\gamma}{16}\left(1 - \frac{1}{m_1}\right), \quad AC_1 = \frac{1}{4}, \quad BC_2 = \frac{1}{4m_1}. \tag{11}$$

By substituting A, B, C_1, C_2 solved in terms of only γ and m_1 into (9) we obtain the desired solution.

We now show that conditions (10) on the physical parameters are naturally imposed when one assumes a special kind of "radial" solution of the form

$$u(r, \theta) = \frac{g(r)}{c_3 r}, \quad \varphi_1(r, \theta) = c_1 g(r) \cos\theta, \quad \varphi_2(r, \theta) = c_2 g(r) \sin\theta, \tag{12}$$

where g is an arbitrary function of r that vanishes at infinity and c_1, c_2, c_3 are arbitrary real constants. These solutions are natural in the sense that $\varphi_{1,x} + \varphi_{2,y}$ remains a purely radial function. From (3) and (4) it follows that $c_1 = m_1 c_2$ and $g(r)$ satisfies

$$c_1 \left[g''(r) + \left(\frac{g(r)}{r} \right)' \right] - \frac{1}{c_3^2} \left(\frac{g^2(r)}{r^2} \right)' = 0.$$

Solving this equation gives us

$$g(r) = \frac{2 c_1 c_3^2 r}{1 + K r^2},$$

where K is an arbitrary constant. When $g(r)$ is substituted into (12), the corresponding expressions are as follows:

$$u(r, \theta) = \frac{2 c_1 c_3}{1 + K r^2}, \quad \varphi_1(r, \theta) = \frac{2 c_1^2 c_3^2 r}{1 + K r^2} \cos\theta,$$
$$\varphi_2(r, \theta) = \frac{1}{m_1} \frac{2 c_1^2 c_3^2 r}{1 + K r^2} \sin\theta. \tag{13}$$

Eq. (13) defines a solution of the GDS system provided that the parameters satisfy $K = \gamma c_1^2 c_3^2(1-1/m_1)/4$ in addition to (10). If we return to the Cartesian coordinates, we obtain a time independent solution of the (HEE) GDS system, with $\mu{=}2c_1c_3$ and $A{=}\gamma(1-1/m_1)/16$,

$$u(x, y) = \frac{\mu}{1 + A\mu^2(x^2 + y^2)}, \quad \varphi_1(x, y) = \frac{\mu^2 x}{2[1 + A\mu^2(x^2 + y^2)]},$$
$$\varphi_2(x, y) = \frac{\mu^2 y}{2m_1[1 + A\mu^2(x^2 + y^2)]} \tag{14}$$

provided that the conditions given in (10) are satisfied. We point out that the solution given in (14) can be obtained from the solution given in (9) through the scale transformation (6).

For the sake of brevity, we rewrite the time independent solution of the (HEE) GDS system in the following form:

$$v_A(x, y) = \mu/[1 + A\mu^2(x^2 + y^2)], \quad \psi_{1A}(x, y) = \mu x v_A/2,$$
$$\psi_{2A}(x, y) = \mu y v_A/(2m_1). \tag{15}$$

Let us observe that although $v_A \in L^2(\mathbb{R}^2)$ and $\|v_A\|_{L^2} = (\pi/A)^{1/2}$, both $(x^2 + y^2)^{1/2} v_A$ is not an element of $L^2(\mathbb{R}^2)$ and ∇v_A is not an element of $\mathbb{L}^2(\mathbb{R}^2)$. Hence this solution does not fall into the solution spaces considered in [1].

Since the solutions of the GDS system is invariant under the pseudo-conformal transformation,

$$u_A(x, y, t) = \frac{1}{(a + bt)} \exp\left(ib\,\frac{y^2 - x^2}{4(a + bt)}\right) v_A(x', y'),$$
$$\varphi_{1A}(x, y, t) = \frac{1}{(a + bt)}\,\psi_{1A}(x', y'),$$
$$\varphi_{2A}(x, y, t) = \frac{1}{(a + bt)}\,\psi_{2A}(x', y')$$

is also a solution. Moreover, a simple computation shows that $\|u_A\|_2 = (\pi/A)^{1/2}$ but $u_A \notin H^1(\mathbb{R}^2)$. Now if we let $T = -a/b$ when $ab < 0$ and set $\varepsilon = a + bt = -b(T - t)$, then $|u_A(x, y, t)|^2$ simplifies to $\varepsilon^{-2}|v_A(x/\varepsilon, y/\varepsilon)|$. Hence, as $t \to T^-$, $\varepsilon \to 0^+$ and

$$|u_A(t)|^2 \to \frac{\pi}{A}\,\delta_0 \quad \text{in } \mathscr{S}',$$

where δ_0 is the Dirac measure at the origin. Here, although we started out with a radially symmetric solution $v_A(x, y)$, $u_A(x, y, t)$ is no longer radially symmetric. However, $|u_A(x, y, t)|$ is radially symmetric and the maximum value of $|u_A(x, y, t)|^2$ is achieved at the origin and is equal to $\mu^2 b^{-2}(T - t)^{-2}$. As $t \to T^-$, this value will blow-up.

Remark. It is worthwhile to observe that $u_A \notin H^1(\mathbb{R}^2)$ and $(x^2 + y^2)^{(s/2)} u_A \notin L^2(\mathbb{R}^2)$ for $s > \frac{2}{3}$, this fact is in agreement with the results on the global existence of solutions for the elliptic NLS equation ([3, Proposition 3.53], in fact, the pseudo-conformal transformation plays an important role in that argument as well).

4. Asymptotic behaviour of solutions in the EEE case

As remarked before the pseudo-conformal invariance is valid in both HEE and EEE cases of the GDS system. The analysis outlined in the previous section does not carry over to the EEE case since the assumption $\sigma=-1$ was indispensable. Let us note that in [9] the authors have utilized the pseudo-conformal invariance advantageously to deduce a blow-up profile for the elliptic NLS equation.

Here we follow an idea given in [13]. We show that the solutions of (2)–(4) that remain in $\Sigma=\{u\in H^1(\mathbb{R}^2):(x^2+y^2)^{1/2}u\in L^2(\mathbb{R}^2)\}$ also satisfy

$$\|u(t)\|_{L^p}^p\leqslant N(1+|t|)^{2-p}$$

for $t>0$, $p>2$ and where N depends only on u_0.

The key idea is similar to that was shown in [1]: for $m_1>0$ and $\kappa\geqslant\max\{-\gamma\max\{1,1/m_1\},0\}$ we have

$$\kappa+\alpha(\xi)\gamma>0$$

for every $\xi\in\mathbb{R}^2$. Therefore, from (5) it is easy to see that

$$\|\nabla u\|^2_{\mathbb{L}^2(\mathbb{R}^2)}\leqslant\mathscr{H}(u)(t)$$

and by the conservation of mass and energy

$$\|u\|^2_{H^1(\mathbb{R}^2)}\leqslant\mathscr{H}(u_0)+\|u_0\|^2_{L^2(\mathbb{R}^2)}.$$

On the other hand, noting that the terms, after the first two, in $(5)_4$ depend only on $|u|$ and combining this with $\|\nabla|u|\|_{\mathbb{L}^2(\mathbb{R}^2)}\leqslant\|\nabla u\|_{\mathbb{L}^2(\mathbb{R}^2)}$ by Stampacchia's inequality, it follows that

$$\mathscr{H}(|u|)(t)\leqslant\mathscr{H}(u)(t). \tag{16}$$

Using the conservation of energy and (16), we obtain

$$\mathscr{H}(|v|)\left(\frac{c+dt}{a+bt}\right)\leqslant\mathscr{H}(v)\left(\frac{c+dt}{a+bt}\right)\leqslant\mathscr{H}(v)\left(\frac{c}{a}\right).$$

If v is pseudo-conformally transformed to u and $T=(c+dt)/(a+bt)$ as before, we get

$$\begin{aligned}\mathscr{H}(|u|)(T)&=(a+bt)^2\mathscr{H}(|v|)(T)\leqslant(a+bt)^2\mathscr{H}(v)\left(\frac{c}{a}\right)\\&=(d-bT)^{-2}\mathscr{H}(v)\left(\frac{c}{a}\right).\end{aligned}$$

It follows that

$$\mathscr{H}(|u|)(T)\leqslant k(1+|T|)^{-2}.$$

By the Gagliardo–Nirenberg inequalities,

$$\begin{aligned}\|u\|^p_{L^p(\mathbb{R}^2)}&\leqslant C\|\nabla|u|\|^{p-2}_{\mathbb{L}^2(\mathbb{R}^2)}\|u\|^2_{L^2(\mathbb{R}^2)}\leqslant C\mathscr{H}(|u|)^{(p-2)/2}\|u_0\|^2_{L^2(\mathbb{R}^2)}\\&\leqslant N(1+|t|)^{2-p}.\end{aligned}$$

Hence we obtain that when $\sigma = 1$ and $\kappa \geqslant \max\{-\gamma \max\{1, 1/m_1\}, 0\}$ then solutions of (2)–(4) that remain in Σ converge to zero in L^p-norm when $p > 2$.

Remark 1. Clearly, a similar argument carries over to the elliptic–elliptic DS system when $\kappa > \gamma$.

Remark 2. It is also possible to obtain decay estimates for the GDS system without invoking the pseudo-conformal invariance. In fact, the general set-up introduced in [6] is applicable to the present setting with $u_0 \in H^2(\mathbb{R}^2)$, $|x|^2 u_0 \in L^2(\mathbb{R}^2)$, $|x|\nabla u_0 \in L^2(\mathbb{R}^2)$. It can then be shown that the L^∞-norm of the solutions decay algebraically to zero as $t \to \infty$.

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