Transverse linear instability of solitary waves for coupled long-wave-short-wave interaction equations

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Abstract

In this paper, we investigate the transverse linear instability of one-dimensional solitary wave solutions of the coupled system of two-dimensional long-wave-short-wave interaction equations. We show that the one-dimensional solitary waves are linearly unstable to perturbations in the transverse direction if the coefficient of the term associated with transverse effects is negative. This transverse instability condition coincides with the non-existence condition identified in the literature for two-dimensional localized solitary wave solutions of the coupled system.

Keywords: Long wave-short wave equations, transverse instability, solitary waves.
2000 MSC: 35Q55, 35B99

1. Introduction

In this study, we conduct linear transverse instability analysis of the one-dimensional solitary wave solutions of the two dimensional long-wave-short-wave interaction (2D-LSI) equations of the form

$$i\phi_t + \alpha\phi_{xx} = \beta\phi u_x, \tag{1}$$

$$u_{xt} + \gamma u_{yy} = -\beta(|\phi|^2)_x, \tag{2}$$

where subscripts refer to partial derivatives, α, β and γ are real parameters, $(x, y) \in \mathbb{R}^2$ are two spatial variables, $t \in \mathbb{R}^+$ is a time-like variable, $\phi = \phi(x, y, t)$ is a complex-valued function and $u = u(x, y, t)$ is a real-valued function.

The 2D-LSI system arises as a mathematical model in various contexts such as water waves [1], geometric optics [2] and elastic waves [3]. It describes the interaction between the high-frequency and low-frequency waves near the long-wave short-wave resonance where the group speed of short waves is equal to the phase speed of long waves. Moreover, in the derivation of these equations, it is assumed that the waves move primarily in the x-direction and that the variations in the y-direction are more gradual. Thus the

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parameter γ measures the relative magnitude of transverse effects in the $y-$direction for the wave motion essentially in the $x-$direction. The well-posedness of the Cauchy problem associated with the 2D-LSI equations has been established in [2]. Assuming $\alpha, \beta > 0$, it is proved in [4] that the two-dimensional localized solitary wave solutions of (1)-(2) may exist in appropriate function spaces if $\gamma > 0$ and they do not exist if $\gamma < 0$. The choice of the sign of γ clearly determines the underlying structure of the localized travelling wave solutions and hence the cases of $\gamma > 0$ and $\gamma < 0$ define quite distinct nature of the 2D-LSI system. As the 2D-LSI system has one-dimensional (i.e. y-independent) solitary wave solutions too, it seems natural to question the instability of the one-dimensional solitary waves to two-dimensional perturbations. Such a transverse instability analysis of line solitary waves for (1)-(2) is the subject of the present study. In particular we show that the one-dimensional solitary waves are linearly unstable if $\gamma < 0$ with respect to transverse perturbations. In other words, we establish that the condition for the transverse linear instability of one-dimensional solitary waves of (1)-(2) coincides with the condition given in [4] for the nonexistence of two-dimensional solitary waves.

The paper is organized as follows. Section 2 is devoted to one-dimensional solitary wave solutions of (1)-(2). In Section 3, a new coordinate system moving with one-dimensional solitary waves is introduced and a discussion of transverse linear instability of one-dimensional solitary wave solutions is given.

Throughout the paper, as usual, we use the notation $\langle ., . \rangle$ to denote the standard inner product on the Hilbert space $L^2(\mathbb{R})$, defined as $\langle f, g \rangle = \int_{\mathbb{R}} f(x) g^*(x) dx$, and also use $\|.\|_p$ to denote the norm $\|f\|_p = (\int_{\mathbb{R}} |f(x)|^p dx)^{1/p}$ in the $L^p(\mathbb{R})$ space.

2. One-Dimensional Solitary Wave Solutions

We start this section by stating an important property of the 2D-LSI equations: the scaling-invariance property. It says that the system (1)-(2) is invariant under the scaling transformation

$$\phi_\lambda(x, y, t) = \lambda^{3/2} \phi(\lambda x, \lambda^{3/2} y, \lambda^2 t), \qquad u_\lambda(x, y, t) = \lambda u(\lambda x, \lambda^{3/2} y, \lambda^2 t) \tag{3}$$

for $\lambda > 0$. If $y-$dependence is dropped from (1)-(2), then the one-dimensional solutions $\phi(x, y, t) = \Phi(x, t)$ and $u(x, y, t) = U(x, t)$ of the 2D-LSI system satisfy the 1D-LSI equations

$$i\Phi_t + \alpha \Phi_{xx} = \beta \Phi U_x, \tag{4}$$

$$U_{xt} = -\beta (|\Phi|^2)_x. \tag{5}$$

These equations were also derived in various fields of physics to describe the resonant interaction of one-dimensional long waves and short waves [5, 6, 7, 8]. As is well known, the 1D-LSI system (4)-(5) has a localized travelling wave solution of the form

$$\begin{aligned} \Phi(x, t) &= \left(\frac{2\alpha}{\beta^2} \right)^{1/2} \operatorname{sech}(x + t) \exp\{ i[-\frac{x}{2\alpha} + \alpha (1 - \frac{1}{4\alpha^2}) t] \}, \\ U_x(x, t) &= -\frac{2\alpha}{\beta} \operatorname{sech}^2(x + t) \end{aligned} \tag{6}$$

for $\alpha > 0$ [4], which represents a solitary wave moving to the left at speed 1. Using the scaling property (3) with $\lambda = c > 0$, a more general form of the solitary wave solution (6) can be given as follows:

$$\Phi_c = \left(\frac{2\alpha c^3}{\beta^2}\right)^{1/2} \operatorname{sech}[c(x+ct)] \exp\{i[-\frac{cx}{2\alpha} + \alpha(1-\frac{1}{4\alpha^2})c^2 t]\},$$
$$U_{cx} = -\frac{2\alpha c^2}{\beta} \operatorname{sech}^2[c(x+ct)], \tag{7}$$

which represents a solitary wave moving at a constant speed c. The stability of the above solitary waves with respect to small but finite spatial perturbations, i.e. the so-called orbital stability, was investigated by Laurençot [9] and, using the Lyapunov stability analysis, it was shown that they are orbitally stable.

A natural question to then ask, which is the topic of the next section, is whether the one-dimensional solitary waves (7) are unstable to small transverse perturbations in two dimensions. This is sometimes called the transverse instability problem and it may also be thought as a structural instability problem since the 2D-LSI system (1)-(2) is a generalization of the 1D-LSI system (4)-(5).

3. Transverse Instability of Line Solitary Waves

In this section, we discuss the transverse linear instability of the 1D-solitary wave solutions of the 2D-LSI system using the perturbation method applied by Zakharov and Rubenchik [10] to deduce instability of 1D standing wave solutions of the 3D nonlinear Schrödinger equations. Because of the scale invariance property of the 2D-LSI equations, in the remainder of this study we only investigate transverse instability of the solitary wave solutions of the form (6). For convenience, we select a new coordinate system (X, Y, T) defined by $X = x + t$, $Y = y$, $T = t$, which moves together with the unperturbed 1D-solitary wave at speed 1. The 2D-LSI system (1)-(2) is then transformed into the new coordinate system and the resulting equations are

$$i(\phi_t + \phi_x) + \alpha\phi_{xx} = \beta\phi u_x, \tag{8}$$
$$u_{xt} + u_{xx} + \gamma u_{yy} = -\beta(|\phi|^2)_x, \tag{9}$$

where we have replaced the letters X, Y and T by x, y and t, respectively, for the convenience of presentation. Transformation of (6) into the new coordinate system yields

$$\Phi(x,t) = \left(\frac{2\alpha}{\beta^2}\right)^{1/2} \operatorname{sech} x \exp\{i[-\frac{x}{2\alpha} + \alpha(1+\frac{1}{4\alpha^2})t]\}, \qquad U_x(x,t) = -\frac{2\alpha}{\beta}\operatorname{sech}^2 x, \tag{10}$$

the transverse instability of which is to be examined in detail.

From now on, we focus on issues concerning the transverse linear instability of the one-dimensional solutions (10) of (8)-(9). For this aim we first write a perturbed solution of (8)-(9) in the form

$$\phi(x,y,t) = \Phi(x,t) + \widetilde{\psi}(x,y,t), \qquad u(x,y,t) = U(x,t) + \widetilde{v}(x,y,t), \tag{11}$$

where $\widetilde{\psi}(x,y,t)$ and $\widetilde{v}(x,y,t)$ denote transverse weak perturbations. Substituting the perturbed solution (11) into the system (8)-(9) and then linearizing the resulting equations with respect to $\widetilde{\psi}(x,y,t)$ and $\widetilde{v}(x,y,t)$, we arrive at a set of linear coupled equations for $\widetilde{\psi}(x,y,t)$ and $\widetilde{v}(x,y,t)$:

$$i(\widetilde{\psi}_t+\widetilde{\psi}_x)+\alpha\widetilde{\psi}_{xx}=\beta(\Phi\widetilde{v}_x+U_x\widetilde{\psi}) \tag{12}$$

$$\widetilde{v}_{xt}+\widetilde{v}_{xx}+\gamma\widetilde{v}_{yy}=-\beta(\Phi\widetilde{\psi}^*+\Phi^*\widetilde{\psi})_x, \tag{13}$$

where $*$ denotes the complex conjugate of the related quantity. In order to simplify the presentation we write $\widetilde{\psi}$ and $\widetilde{v}$ in the form

$$\widetilde{\psi}(x,y,t)=\left(\frac{2\alpha}{\beta^2}\right)^{1/2}\psi(x,y,t)\exp\{i[-\frac{x}{2\alpha}+\alpha(1+\frac{1}{4\alpha^2})t]\},\qquad \widetilde{v}(x,y,t)=\frac{2\alpha}{\beta}\,v(x,y,t).$$

In terms of $\psi(x,y,t)$ and $v(x,y,t)$, the system (12)-(13) becomes

$$i\psi_t-\alpha\psi+\alpha\psi_{xx}=2\alpha[v_x\ R(x)-\psi\ R^2(x)] \tag{14}$$

$$v_{xt}+v_{xx}+\gamma v_{yy}=-[(\psi+\psi^*)\ R(x)]_x, \tag{15}$$

where, for convenience, we have used the notation $R(x)=\text{sech}\ x$ as we do henceforth. If we decompose $\psi(x,y,t)$ into its real and imaginary parts by writing $\psi(x,y,t)=p(x,y,t)+iq(x,y,t)$, then (14)-(15) takes the following form

$$-p_t=\alpha q_{xx}+\alpha(2\ R^2(x)-1)q, \tag{16}$$

$$q_t=\alpha p_{xx}+\alpha(2\ R^2(x)-1)p-2\alpha\ R(x)v_x, \tag{17}$$

$$-v_{xt}=v_{xx}+\gamma v_{yy}+2(R(x)\ p)_x \tag{18}$$

for the real-valued perturbations p, q and v.

We now assume that the real-valued perturbations are of the form

$$p(x,y,t)=P(x)e^{iky+\Omega t}+P^*(x)e^{-iky+\Omega^* t} \tag{19}$$

$$q(x,y,t)=Q(x)e^{iky+\Omega t}+Q^*(x)e^{-iky+\Omega^* t} \tag{20}$$

$$v(x,y,t)=V(x)e^{iky+\Omega t}+V^*(x)e^{-iky+\Omega^* t}, \tag{21}$$

where $k\in\mathbb{R}$, $\Omega\in\mathbb{C}$ and $P(x)$, $Q(x)$ and $V(x)$ are complex-valued functions. It should be noted that for transverse instability of the one-dimensional solitary waves, the parameter Ω must have positive real part: $\text{Re}(\Omega)>0$. Substitution of (19)-(21) into (16)-(18) gives rise to a set of coupled ordinary differential equations for P, Q and V

$$\mathcal{L}Q=\Omega P, \tag{22}$$

$$\mathcal{L}P+2\alpha RV'=-\Omega Q, \tag{23}$$

$$-V''+\gamma k^2V-2(RP)'=\Omega V', \tag{24}$$

where the prime denotes differentiation with respect to x and $\mathcal{L}$ is the linear self-adjoint operator defined by $\mathcal{L}=\alpha[-\frac{d^2}{dx^2}+1-2R^2(x)]$. The next step is to assume that both a candidate solution $(P(x),\ Q(x),\ V(x))$ of (22)-(24) and the parameter $\Omega(k)$ can be written as an asymptotic power series expansion in k

$$P(x) = P_0(x) + kP_1(x) + k^2P_2(x) + \cdots\ , \tag{25}$$
$$Q(x) = Q_0(x) + kQ_1(x) + k^2Q_2(x) + \cdots\ , \tag{26}$$
$$V(x) = V_0(x) + kV_1(x) + k^2V_2(x) + \cdots\ , \tag{27}$$
$$\Omega = k\Omega_1 + k^2\Omega_2 + \cdots\ . \tag{28}$$

Substituting (25)-(28) into (22)-(24) and then equating terms of the same order in k we obtain a hierarchical set of ordinary differential equations. In the remaining part of this section we solve explicitly the equations corresponding to the first three orders of the hierarchy.

The lowest-order equations of the hierarchy are obtained in the form

$$\mathcal{L}Q_0 = 0, \tag{29}$$
$$\mathcal{L}P_0 = -2\alpha RV_0', \tag{30}$$
$$V_0'' + 2(RP_0)' = 0. \tag{31}$$

If we integrate (31) once and use the boundary conditions at infinity, i.e. $V'(x)\ \to\ 0$ as $x\ \to\ \pm\infty$, we get $V_0'(x) = -2R(x)P_0(x)$. Using this result in (30) we obtain $\mathcal{N}P_0 = 0$ where $\mathcal{N}$ is the linear operator defined by $\mathcal{N} = \alpha[-\dfrac{d^2}{dx^2} + 1 - 6R^2(x)]$. The functions $R(x)$ and $R'(x)$ are the kernels of the operators $\mathcal{L}$ and $\mathcal{N}$, respectively, and the solutions of the equations $\mathcal{L}Q_0 = 0$ and $\mathcal{N}P_0 = 0$ are of the form

$$Q_0(x) = a_0\mathrm{sech}x = a_0R(x), \qquad P_0(x) = -b_0\tanh x\ \mathrm{sech}x = b_0R'(x),$$

where a_0 and b_0 are arbitrary constants. It follows from the relations $V_0' = -2RP_0$ and $P_0 = b_0R'$ that $V_0(x) = -b_0R^2(x) + d_0$ where d_0 is an arbitrary constant.

The first-order equations in k are

$$\mathcal{L}Q_1 = \Omega_1P_0, \tag{32}$$
$$\mathcal{L}P_1 + 2\alpha RV_1' = -\Omega_1Q_0, \tag{33}$$
$$V_1'' + 2(RP_1)' = -\Omega_1V_0'. \tag{34}$$

A solution of (32) is given by

$$Q_1(x) = -\frac{b_0\Omega_1}{2\alpha}xR(x) + a_1R(x) \tag{35}$$

where a_1 is an arbitrary constant. Integrating (34) once and using the boundary conditions at infinity, i.e. $V_1'(x)\ \to\ 0$ as $x\ \to\ \pm\infty$, we get

$$V_1'(x) = -2R(x)P_1(x) + b_0\Omega_1R^2(x) \tag{36}$$

and $d_0 = 0$. Substitution of this result into (33) gives

$$\mathcal{N}P_1(x) = -a_0\Omega_1R(x) - 2\alpha b_0\Omega_1R^3(x). \tag{37}$$

A solution of this equation is given as follows:

$$P_1(x) = \frac{a_0\Omega_1}{2\alpha}[xR'(x) + R(x)] + \frac{b_0\Omega_1}{2}R(x) + b_1 R'(x) \tag{38}$$

where b_1 is an arbitrary constant. Since Q_0 and P_0 are the kernels of $\mathcal{L}$ and $\mathcal{N}$, the right-hand sides of (32) and (37) must be orthogonal to Q_0 and P_0, respectively. That is, the orthogonality conditions $\langle \mathcal{L}Q_1, Q_0\rangle = 0$ and $\langle \mathcal{N}P_1, P_0\rangle = 0$ hold. Noting that $\langle R, R'\rangle = 0$ and $\langle R^3, R'\rangle = 0$, we address this issue through a simple calculation

$$\begin{aligned}
\langle \mathcal{L}Q_1, Q_0\rangle &= \langle Q_1, \mathcal{L}Q_0\rangle = \Omega_1 a_0 b_0 \langle R, R'\rangle = 0,\\
\langle \mathcal{N}P_1, P_0\rangle &= \langle P_1, \mathcal{N}P_0\rangle = -\Omega_1 a_0 b_0 \langle R, R'\rangle - 2\alpha\Omega_1 b_0^2 \langle R^3, R'\rangle = 0.
\end{aligned}$$

The second-order equations in k are

$$\mathcal{L}Q_2 = \Omega_1 P_1 + \Omega_2 P_0, \tag{39}$$

$$\mathcal{L}P_2 + 2\alpha R V_2' = -(\Omega_1 Q_1 + \Omega_2 Q_0), \tag{40}$$

$$V_2'' - \gamma V_0 + 2(RP_2)' = -(\Omega_1 V_1' + \Omega_2 V_0'). \tag{41}$$

Substitution of (35), (36) and (38) into (39)-(40) gives

$$\mathcal{L}Q_2 = \Omega_1\left(\frac{a_0\Omega_1}{2\alpha}(xR)' + \frac{b_0\Omega_1}{2}R\right) + (\Omega_1 b_1 + \Omega_2 b_0)R', \tag{42}$$

$$\mathcal{L}P_2 + 2\alpha R V_2' = \Omega_1^2 \frac{b_0}{2\alpha}(xR) - (\Omega_1 a_1 + \Omega_2 a_0)R, \tag{43}$$

$$V_2'' + \gamma b_0 R^2 + 2(RP_2)' = \frac{a_0\Omega_1^2}{\alpha}R(xR)' + (\Omega_1 b_1 + \Omega_2 b_0)(R^2)'. \tag{44}$$

Equation (42) can be solved if the right-hand side is orthogonal to the kernel of $\mathcal{L}$, i.e. to Q_0. This orthogonality condition leads to

$$\langle \mathcal{L}Q_2, R\rangle = \frac{\Omega_1^2}{2}\left(\frac{a_0}{2\alpha} + b_0\right)\|R\|_2^2 = 0, \tag{45}$$

where (44) is used. For $\Omega_1 \neq 0$, this result says that the first orthogonality condition $\langle \mathcal{L}Q_2, Q_0\rangle = 0$ holds provided that $\frac{a_0}{2\alpha} + b_0 = 0$ which gives $a_0 = -2\alpha b_0$ and eliminates a_0 from the problem. To find the restriction imposed by the second orthogonality condition $\langle \mathcal{N}P_2, P_0\rangle = 0$, using (43) we first compute

$$\langle \mathcal{L}P_2, R'\rangle = -\alpha\langle V_2', (R^2)'\rangle - \frac{b_0\Omega_1^2}{4\alpha}\|R\|_2^2. \tag{46}$$

Then, using integration by parts and (44), one gets

$$\alpha\langle V_2'', R^2\rangle = -\alpha\langle V_2', (R^2)'\rangle = -\gamma\alpha b_0\|R\|_4^4 - \frac{3}{2}\alpha b_0\Omega_1^2\|R\|_4^4 + 4\alpha\langle R^2, R'P_2\rangle. \tag{47}$$

When we substitute (47) into (46), we obtain

$$\langle \mathcal{N}P_2, R'\rangle = -\gamma\alpha b_0\|R\|_4^4 - \frac{3}{2}\alpha b_0\Omega_1^2\|R\|_4^4 - \frac{b_0\Omega_1^2}{4\alpha}\|R\|_2^2. \tag{48}$$

As a result, the second orthogonality condition holds provided that

$$4\gamma\alpha^2\|R\|_4^4+\Omega_1^2\left(6\alpha^2\|R\|_4^4+\|R\|_2^2\right)=0 \tag{49}$$

with $\Omega_1\neq 0$. Using $\|R\|_2^2=\int_{\mathbb{R}}\operatorname{sech}^2x\mathrm{d}x=2, \qquad \|R\|_4^4=\int_{\mathbb{R}}\operatorname{sech}^4x\mathrm{d}x=\frac{4}{3}$ in (49) and solving the resulting equation for Ω_1^2 gives

$$\Omega_1^2=-\frac{8\alpha^2\gamma}{12\alpha^2+3}. \tag{50}$$

This expression shows that the eigenvalues $\pm\Omega_1$ are purely real if $\gamma<0$. Since the parameter γ measures transverse effects, the unperturbed solitary wave solution is said to be linearly unstable against transverse perturbations in the case of negative dispersion ($\gamma<0$).

At this point, it is interesting to note that the condition given here for the transverse linear instability of one-dimensional solitary waves of the 2D-LSI equations is the same as that imposed in [4] for the nonexistence of two-dimensional localized solutions of the 2D-LSI equations. Besides providing an explanation for this apparent coincidence, one important question about the 2D-LSI equations still remains open for investigation. This issue is to extend our instability analysis to the regime in which transverse perturbations become large, that is, to find out for what values of γ the one-dimensional solitary wave solutions of the 2D-LSI equations are nonlinearly unstable with respect to transverse perturbations.

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