Physica Scripta. Vol. 58, 9–14, 1998

Nonlinear Transverse Waves in a Generalized Elastic Solid and the Complex Modified Korteweg–de Vries Equation

H. A. Erbay*

Department of Mathematics, Faculty of Science, Istanbul Technical University, Maslak 80626, Istanbul, Turkey

Received August 4, 1997; accepted in revised form January 12, 1998

PACS Ref: 03.40.Kf, 43.25.Rq, 62.30. + d

Abstract

By using an asymptotic expansion technique based on the Lagrangian density formulation, it is demonstrated that the nonlinear interaction between two transverse waves propagating in a generalized elastic solid is described asymptotically by the complex modified Korteweg–de Vries (CMKdV) equation. Also, the CMKdV equation is shown to be Hamiltonian and its solitary wave solutions are investigated by means of a modified Hirota method.

1. Introduction

Various theories have been proposed in recent years to incorporate the internal, discrete, structure of matter into the classical elasticity model. These theories take different names depending on which aspect of continuum has been chosen as a starting point. Higher order gradients, nonlocal particle interactions, polyatomic structure and local intrinsic rotations are some of these aspects, among others. From the viewpoint of the wave phenomena, it is interesting that all these theories, contrary to the classical elasticity theory, allow us to observe *dispersive* wave propagation in linear approximation. This feature of the models, i.e. their dispersive character, may give rise to coherent structures such as solitary waves if nonlinearity is included.

In this study one of the above-mentioned generalized continuum theories, i.e. micropolar elasticity theory which takes into account the local rigid microrotations of material particles, will be considered. The theory of micropolar elasticity is concerned with an elastic medium whose constituents, the so-called material points, are allowed to rotate independently without stretch. Hence, the motion of the material points of such a medium will have three additional degrees of freedom associated with local micro-rigid rotations. The fundamental equations of a micropolar elastic medium then contain coupled microrotation and displacement fields [1]. The theory has been proposed to study the mechanical behavior of materials with fibrous and elongated grains (e.g., dumb-bell types of grains). Also molecular crystals represent a potential field of application of the theory. A deformable lattice equipped, at each node, with molecules performing rigid-body rotational motions about the center of mass provides a fairly good description of a micropolar elastic medium. For instance, such a model has been proposed in [2] to study the dynamics of certain molecular crystals, where the potassium nitrate crystal is selected as an example material.

In [3], using the reductive perturbation method Erbay and Suhubi showed that the nonlinear interaction of the long transverse (displacement or microrotation) waves propagating in a weakly nonlinear micropolar elastic solid is described asymptotically by the complex modified Korteweg–de Vries (CMKdV) equation. In [4], they found that the CMKdV equation does not have the Painlevé property. Similarity solutions of the equation are studied in [5] recently. The same equation has been also proposed as a model for the nonlinear evolution of plasma waves [6] and derived to describe the propagation of transverse waves in a molecular chain model [7].

In the first part of the present study, the CMKdV equation is rederived for the interaction of two transverse waves propagating in a weakly nonlinear micropolar elastic solid. Our derivation follows a method developed in [8] and [9] (Chapter 5) for ion acoustic waves in an unmagnetized plasma. In this method, the Lagrangian density, instead of operating on the equations of motion, is expanded in terms of a small parameter. The existence of a variational formulation is not crucial in obtaining a nonlinear evolution equation; the same result could be obtained by considering the equations of motion and using the reductive perturbation method as in [3]. However, the knowledge of the Lagrangian density makes the derivation mathematically simpler and the calculations are greatly simplified. Moreover, the present method yields the Lagrangian density for the nonlinear evolution equation, i.e. for the CMKdV equation, in a straightforward fashion. In Section 3 we show that the CMKdV equation is a Hamiltonian system. In Section 4, using a modified Hirota method, solitary wave solutions of the CMKdV equation are studied.

2. A derivation of the equation

The purpose of the present section is to attempt to apply the method suggested in [9] to the governing equations of weakly nonlinear micropolar elastic solids. The model of the present work is one dimensional and based on the Lagrange density function $\mathscr{L}$ defined by

$$\mathscr{L} = T - \Sigma \tag{2.1}$$

* E-mail: erbay@sariyer.cc.itu.edu.tr

It gives the following results as the Euler–Lagrange equations

$$\delta u_1^{(1)}:\ u_1^{(1)} = 0, \tag{2.8}$$

$$\delta \varphi_1^{(1)}:\ \varphi_1^{(1)} = 0, \tag{2.9}$$

$$\delta u_2^{(1)},\ \delta \varphi_3^{(1)}:\ \varphi_3^{(1)} = \frac{1}{2}\frac{\partial u_2^{(1)}}{\partial \xi}, \tag{2.10}$$

$$\delta u_3^{(1)},\ \delta \varphi_2^{(1)}:\ \varphi_2^{(1)} = -\frac{1}{2}\frac{\partial u_3^{(1)}}{\partial \xi}. \tag{2.11}$$

Here the inequalities $\kappa \neq 0$ and $(\lambda + \mu + \kappa/2) \neq 0$ [3] are used. As seen from these relations, all the first-order longitudinal components are zero, whereas the transverse microrotation components are known in terms of two transverse displacement components, namely, $\partial u_2^{(1)}/\partial \xi$ and $\partial u_3^{(1)}/\partial \xi$, which will be determined from higher-order perturbation problems.

If the above first-order results (2.8)–(2.11) are used, the second-order Lagrange density function becomes identically zero, i.e.

$$\mathscr{L}^{(2)} = 0$$

and consequently it does not give any new results.

Using the first-order results (2.8)–(2.11), the third-order Lagrange density function is calculated as

$$\begin{aligned}
\mathscr{L}^{(3)} = &-\frac{1}{2}\left(\lambda + \mu + \frac{\kappa}{2}\right)\left(\frac{\partial u_1^{(2)}}{\partial \xi}\right)^2 \\
&-\frac{\kappa}{4}\left[\left(\frac{\partial u_2^{(2)}}{\partial \xi}\right)^2 + \left(\frac{\partial u_3^{(2)}}{\partial \xi}\right)^2\right] \\
&-\kappa[(\varphi_1^{(2)})^2 + (\varphi_2^{(2)})^2 + (\varphi_3^{(2)})^2] \\
&+\kappa\left(\varphi_3^{(2)}\frac{\partial u_2^{(2)}}{\partial \xi} - \varphi_2^{(2)}\frac{\partial u_3^{(2)}}{\partial \xi}\right) \\
&-\rho_0 c\left(\frac{\partial u_2^{(1)}}{\partial \tau}\frac{\partial u_2^{(1)}}{\partial \xi} + \frac{\partial u_3^{(1)}}{\partial \tau}\frac{\partial u_3^{(1)}}{\partial \xi}\right) \\
&-\frac{1}{8}(\gamma - \rho_0 J_0 c^2)\left[\left(\frac{\partial^2 u_2^{(1)}}{\partial \xi^2}\right)^2 + \left(\frac{\partial^2 u_3^{(1)}}{\partial \xi^2}\right)^2\right] \\
&-\frac{1}{2}\left(\frac{a_2}{4} + a_3 + a_5\right)\left[\left(\frac{\partial u_2^{(1)}}{\partial \xi}\right)^2 + \left(\frac{\partial u_3^{(1)}}{\partial \xi}\right)^2\right]\frac{\partial u_1^{(2)}}{\partial \xi}.
\end{aligned} \tag{2.12}$$

The Euler–Lagrange equations corresponding to $\mathscr{L}^{(3)}$ are obtained as follows

$$\begin{aligned}
\delta u_2^{(1)}:\ &2\rho_0 c\frac{\partial}{\partial \tau}\left(\frac{\partial u_2^{(1)}}{\partial \xi}\right) + \left(\frac{a_2}{4} + a_3 + a_5\right)\frac{\partial}{\partial \xi} \\
&\times\left(\frac{\partial u_2^{(1)}}{\partial \xi}\frac{\partial u_1^{(2)}}{\partial \xi}\right) - \frac{1}{4}(\gamma - \rho_0 J_0 c^2)\frac{\partial^3}{\partial \xi^3}\left(\frac{\partial u_2^{(1)}}{\partial \xi}\right) = 0
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\delta u_3^{(1)}:\ &2\rho_0 c\frac{\partial}{\partial \tau}\left(\frac{\partial u_3^{(1)}}{\partial \xi}\right) + \left(\frac{a_2}{4} + a_3 + a_5\right)\frac{\partial}{\partial \xi} \\
&\times\left(\frac{\partial u_3^{(1)}}{\partial \xi}\frac{\partial u_1^{(2)}}{\partial \xi}\right) - \frac{1}{4}(\gamma - \rho_0 J_0 c^2)\frac{\partial^3}{\partial \xi^3}\left(\frac{\partial u_3^{(1)}}{\partial \xi}\right) = 0
\end{aligned} \tag{2.14}$$

$$\delta u_1^{(2)}:\ \frac{\partial u_1^{(2)}}{\partial \xi} = -\frac{\left(\frac{a_2}{4} + a_3 + a_5\right)}{2\left(\lambda + \mu + \frac{\kappa}{2}\right)}\left[\left(\frac{\partial u_2^{(1)}}{\partial \xi}\right)^2 + \left(\frac{\partial u_3^{(1)}}{\partial \xi}\right)^2\right], \tag{2.15}$$

$$\delta \varphi_1^{(2)}:\ \varphi_1^{(2)} = 0, \tag{2.16}$$

$$\delta u_2^{(2)},\ \delta \varphi_3^{(2)}:\ \varphi_3^{(2)} = \frac{1}{2}\frac{\partial u_2^{(2)}}{\partial \xi}, \tag{2.17}$$

$$\delta u_3^{(2)},\ \delta \varphi_2^{(2)}:\ \varphi_2^{(2)} = -\frac{1}{2}\frac{\partial u_3^{(2)}}{\partial \xi}. \tag{2.18}$$

Now, using eq. (2.15) in eqs (2.13) and (2.14) in order to eliminate the higher-order term $\partial u_1^{(2)}/\partial \xi$, the Euler–Lagrange equations corresponding to $\delta u_2^{(1)}$ and $\delta u_3^{(1)}$, eqs (2.13) and (2.14), can be written as

$$\begin{aligned}
&\frac{\partial}{\partial \tau}\left(\frac{\partial u_2^{(1)}}{\partial \xi}\right) + a\frac{\partial}{\partial \xi}\left\{\left[\left(\frac{\partial u_2^{(1)}}{\partial \xi}\right)^2 + \left(\frac{\partial u_3^{(1)}}{\partial \xi}\right)^2\right]\frac{\partial u_2^{(1)}}{\partial \xi}\right\} \\
&\quad + b\frac{\partial^3}{\partial \xi^3}\left(\frac{\partial u_2^{(1)}}{\partial \xi}\right) = 0,
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
&\frac{\partial}{\partial \tau}\left(\frac{\partial u_3^{(1)}}{\partial \xi}\right) + a\frac{\partial}{\partial \xi}\left\{\left[\left(\frac{\partial u_2^{(1)}}{\partial \xi}\right)^2 + \left(\frac{\partial u_3^{(1)}}{\partial \xi}\right)^2\right]\frac{\partial u_3^{(1)}}{\partial \xi}\right\} \\
&\quad + b\frac{\partial^3}{\partial \xi^3}\left(\frac{\partial u_3^{(1)}}{\partial \xi}\right) = 0.
\end{aligned} \tag{2.20}$$

where the coefficients a and b are defined as

$$a = -\frac{\left(\frac{a_2}{4} + a_3 + a_5\right)^2}{4\rho_0 c\left(\lambda + \mu + \frac{\kappa}{2}\right)}, \quad b = -\frac{\gamma - \rho_0 J_0 c^2}{8\rho_0 c}.$$

The other Euler–Lagrange equations, eqs (2.16)–(2.18), give some relations between higher-order terms. Note that the interaction between the first-order transverse components $\partial u_2^{(1)}/\partial \xi$ and $\partial u_3^{(1)}/\partial \xi$ arise through the second-order longitudinal component $\partial u_1^{(2)}/\partial \xi$ [see eqs (2.13)–(2.15)]. For convenience, the spatial derivatives of the first-order transverse displacement components are shown as follows

$$\Phi = \frac{\partial u_2^{(1)}}{\partial \xi}, \quad \Psi = \frac{\partial u_3^{(1)}}{\partial \xi}.$$

Thus eqs (2.19) and (2.20) take the following form:

$$\frac{\partial \Phi}{\partial \tau} + a\frac{\partial}{\partial \xi}[(\Phi^2 + \Psi^2)\Phi] + b\frac{\partial^3 \Phi}{\partial \xi^3} = 0, \tag{2.21}$$

$$\frac{\partial \Psi}{\partial \tau} + a\frac{\partial}{\partial \xi}[(\Phi^2 + \Psi^2)\Psi] + b\frac{\partial^3 \Psi}{\partial \xi^3} = 0. \tag{2.22}$$

These two coupled nonlinear equations describe the interaction of two linearly polarized transverse waves. For some special cases, i.e. for $\Psi = 0$ or $\Phi = 0$ or $\Phi = \Psi$ which correspond to the 0°, 90° and 45° polarizations, respectively, these two coupled equations reduce to the single modified Korteweg–de Vries (MKdV) equation. If a new complex quantity is defined in terms of Φ and Ψ as $w = \Phi + i\Psi$, the above two coupled equations which are called the vector MKdV equations sometimes reduce to the following single

4. Modified Hirota method

In recent years Hirota's direct method has played a key role in identifying integrable nonlinear dynamical systems and it has been applied successfully to a number of nonlinear evolution equations [11]. In [12] and [13] a modified Hirota method has been applied to the two coupled nonlinear Schrödinger equations with a linear birefringence term in order to obtain a new family of solitary wave solutions. This new family of solitary waves reduces to the classical solitary wave solution of the single nonlinear Schrödinger equation when one of the complex amplitudes vanishes or two complex amplitudes are equal to each other. The unusual characteristic about the modified Hirota method in comparison to the classical one is that it begins with a reduction of the original partial differential equations to the ordinary differential equations by means of a travelling-wave transformation. Then it involves a series expansion of dependent variables. As in the classical one, one of the most important aspects of the modified Hirota method is that it may be possible to truncate the series expansion at a finite number of terms. In this section we study special solutions of the CMKdV equation (2.23) by using the modified Hirota method developed in [12] and [13].

We begin by introducing the following simple travelling-wave transformation

$$\Phi(\xi, \tau) = (b/a)^{1/2}\phi(\zeta), \qquad \Psi(\xi, \tau) = (b/a)^{1/2}\psi(\zeta) \tag{4.1}$$

where ϕ and ψ are real functions of $\zeta = \xi - v_0\tau$ alone and v_0 is a real constant. The next step is to substitute the ansatz (4.1) into eq. (2.23) and use Hirota's approach to solve the resulting ordinary differential equations.

Now substituting the ansatz (4.1) into the CMKdV equation (2.23) we obtain

$$\phi_{\zeta\zeta} - C^2\phi + (\phi^2 + \psi^2)\phi = 0, \qquad \psi_{\zeta\zeta} - C^2\psi + (\phi^2 + \psi^2)\psi = 0 \tag{4.2}$$

where $C^2 = v_0/b$. To solve the system (4.2) we apply the modified method by assuming the following forms of ϕ and ψ in terms of real functions $f(\zeta)$, $g(\zeta)$ and $h(\zeta)$:

$$\phi(\zeta) = g(\zeta)/f(\zeta), \quad \psi(\zeta) = h(\zeta)/f(\zeta) \tag{4.3}$$

where $f(\zeta)$ satisfies the relation

$$\phi^2 + \psi^2 = 2(\ln f)_{\zeta\zeta}. \tag{4.4}$$

Using eqs (4.2), (4.3) and (4.4) we obtain a set of three coupled equations for f, g and h, namely

$$fg_{\zeta\zeta} + gf_{\zeta\zeta} - 2f_\zeta g_\zeta = C^2 fg, \qquad fh_{\zeta\zeta} + hf_{\zeta\zeta} - 2f_\zeta h_\zeta = C^2 fh, \qquad g^2 + h^2 = 2f^2(\ln f)_{\zeta\zeta}. \tag{4.5}$$

We look for solutions to these equations in the form of power series in a parameter δ which is introduced simply to keep track of the terms in the expansion

$$f = 1 + \sum_{n=1}^{\infty} f^{(n)}\delta^n, \quad g = \sum_{n=1}^{\infty} g^{(n)}\delta^n, \qquad h = \sum_{n=1}^{\infty} h^{(n)}\delta^n. \tag{4.6}$$

Solutions such that the series terminates after a finite number of terms will now be looked for. Substitution of eq. (4.6) into the system (4.5) gives the following first-order problem

$$f^{(1)}_{\zeta\zeta} = 0, \quad g^{(1)}_{\zeta\zeta} - C^2 g^{(1)} = 0, \quad h^{(1)}_{\zeta\zeta} - C^2 h^{(1)} = 0$$

for which a solution is

$$f^{(1)} = 0, \quad g^{(1)} = 2\sqrt{2}\,A \exp(\theta), \quad h^{(1)} = 2\sqrt{2}\,B \exp(\theta) \tag{4.7}$$

where

$$\theta = C(\zeta - \zeta_0)$$

in which ζ_0 is an arbitrary constant.

For second order in δ, taking into account eq. (4.7), we find

$$f^{(2)}_{\zeta\zeta} = 4(A^2 + B^2)\exp(2\theta),$$

$$g^{(2)}_{\zeta\zeta} - C^2 g^{(2)} = 0, \quad h^{(2)}_{\zeta\zeta} - C^2 h^{(2)} = 0$$

which gives

$$f^{(2)} = \sigma \exp(2\theta), \quad g^{(2)} = 0, \quad h^{(2)} = 0 \tag{4.8}$$

where

$$\sigma = (A^2 + B^2)/C^2.$$

We now proceed to the third-order problem. Using the results (4.7) and (4.8) previously obtained, simple but tedious algebra then shows that the right-hand side of the third-order problem is identically zero so that we can choose $f^{(3)} = g^{(3)} = h^{(3)} = 0$. Furthermore, at this point we assume that the series can be truncated, that is, all the higher order terms can be set to zero. Then putting $\delta = 1$ we have an exact solution of eq. (4.5) in the form

$$f(\zeta) = 1 + \sigma \exp(2\theta), \quad g(\zeta) = 2\sqrt{2}\,A \exp(\theta), \qquad h(\zeta) = 2\sqrt{2}\,B \exp(\theta). \tag{4.9}$$

Substituting eq. (4.9) into eq. (4.5) verifies that this is indeed a solution, thus justifying our assumption.

If the following condition is assumed

$$A^2 + B^2 = C^2 \tag{4.10}$$

in which $\sigma = 1$, we obtain from eqs (4.9) and (4.3)

$$\phi(\zeta) = \sqrt{2}\,A \operatorname{sech}[C(\zeta - \zeta_0)], \qquad \psi(\zeta) = \sqrt{2}\,B \operatorname{sech}[C(\zeta - \zeta_0)]. \tag{4.11}$$

Using eq. (4.1) and returning to the original variables we finally have

$$\Phi(\xi, \tau) = \left(\frac{2b}{a}\right)^{1/2} A \operatorname{sech}[C(\xi - v_0\tau - \zeta_0)], \qquad \Psi(\xi, \tau) = \left(\frac{2b}{a}\right)^{1/2} B \operatorname{sech}[C(\xi - v_0\tau - \zeta_0)]. \tag{4.12}$$

In order to get some feeling as to the meaning of this result, it is somewhat instructive to examine the following two special cases.

Case (*i*): There exist waves with linear polarization in one of the transverse directions only in the medium. That is, either