PULSE WAVES IN PRESTRESSED ARTERIES

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In order to better understand the effect of initial stress in blood flow in arteries, a theoretical analysis of wave propagation in an initially inflated and axially stretched cylindrical thick shell is investigated. For simplicity in the mathematical analysis, the blood is assumed to be an incompressible inviscid fluid while the arterial wall is taken to be an isotropic, homogeneous and incompressible elastic material. Employing the theory of small deformations superimposed on a large initial field the governing differential equations of perturbed solid motions are obtained in cylindrical polar coordinates. Considering the difficulty in obtaining a closed form solution for the field equations, an approximate power series method is utilized. The dispersion relations for the most general case of this approximation and for the thin tube case are thoroughly discussed. The speeds of waves propagating in an unstressed tube are obtained as a special case of our general treatment. It is observed that the speeds of both waves increase with increasing inner pressure and axial stretch.

1. Introduction. Propagation of pulse waves in an initially stressed (or unstressed) elastic, cylindrical tube containing a viscous or inviscid fluid has been a problem of interest since the time of Thomas Young who first studied the pulse waves in human arteries. This subject has received a great deal of interest and activity among the researchers in the last two decades or so. The literature on the subject is so rich that we cannot cite all the contributed works here. The historical perspective of the problem is to be found in the papers by Lambossy (1951) and Skalak (1966) and in the books by Attinger (1964), McDonald (1966) and Fung (1984). Significant investigations of wave motions of a viscous fluid in an elastic tube have been carried out by Morgan and Kiely (1954), Womersley (1957) and Atabek and Lew (1966). These researchers have assumed that the arterial wall is a thin walled isotropic elastic tube. As indicated in their experimental studies, in reality, the artery may have thick walls with viscoelastic and anisotropic properties (Lawton, 1955; Fen, 1957; Bergel, 1961). These characteristics of blood vessels have been taken into account by Mirsky (1967), Atabek (1968), Rubinow and Keller (1971) and more recently by Rachev (1980) and Kuiken (1984).

In all these works either the effects of initial stresses to flow characteristics have been neglected or the artery has been treated as a membrane or a thin

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shell. In order to have an applicable thin shell theory, the ratio of wall thickness to midradius should be less than 1/20. However, even for large arteries this ratio changes between 1/6 and 1/4. Moreover, as pointed out by Demiray (1976) and Fung et al. (1979), the initial stress distribution through the wall is not uniform, it rather changes drastically from the inner surface to the outer surface of the artery. Therefore, the assumption of constant initial stress and associated membrane or shell theories cannot be applied to arterial mechanics.

In the present work, employing the field equations and boundary conditions of a theory so called 'the small deformation superimposed on large initial static deformations', the propagation of a harmonic wave in a cylindrical artery filled with an incompressible fluid is studied. Although blood is known to be a non-Newtonian fluid, for the sake of simplicity in the analysis, it is assumed to be incompressible and inviscid, while the arterial wall is treated as an isotropic and incompressible elastic solid subjected to a large initial deformation. For an axially symmetric perturbed motion, the governing differential equations of the solid body are obtained in cylindrical polar coordinates. Due to variability of initial stress, the coefficients of resulting differential equations are also variable. Considering the difficulty of obtaining a closed form solution to such a set of differential equations, a truncated power series method is presented. After utilizing the boundary conditions the dispersion relation is obtained as a function of inner pressure, axial stretch and the thickness ratio. Employing an approximation similar to one used by Rubinow and Keller (1971), the dispersion relation is obtained for thin tubes with or without initial stresses. For wavelengths large, as compared to the mean radius, it is observed that the wave speeds reduce to the Lamb and Moens-Korteweg (Young) modes. The numerical results further indicate that the wave speeds increase with increasing inner pressure and axial stretch. Although the Lamb mode is almost insensitive to the changes of wave number, the phase velocity corresponding to the Young mode increases with wave number depending on the strength of the inner pressure and axial stretch.

2. Governing Equations and Boundary Conditions. The phenomenon of wave propagation in arteries is caused by the interaction of the blood with its container. The mathematical formulation should, therefore, include the proper field equations and associated boundary (interface) conditions.

2.1. Fluid equations. In general, blood is known to be a non-Newtonian fluid. However, for simplicity in the analysis we shall neglect the effect of viscosity and assume that blood is an incompressible inviscid fluid subjected to a constant uniform pressure $P$. If such an equilibrium state of the fluid is disturbed by a pressure pulse $\tilde{p}(r, z, t)$ created by the left ventricle, a harmonic wave type of flow field will be developed in the blood. Assuming that the initial velocity field vanishes, for axisymmetric motions, the governing field equations
in cylindrical polar coordinates will take the following form:

\[ \bar{p}_{,r} + \bar{p} \bar{u}_{,r} = 0, \quad \bar{p}_{,z} + \bar{p} \bar{v}_{,r} = 0, \quad \bar{u}_{,r} + \bar{u}/r + \bar{v}_{,z} = 0 \]  

(1)

where \( \bar{p} \) is the fluid mass density, \( v_1 = \bar{u}(r, z, t), (v_2 = \bar{v} = 0), v_3 = \bar{w}(r, z, t) \) are the velocity components of the fluid in the radial and axial directions, respectively, and \( \bar{p}(r, z, t) \) is the increment in fluid pressure. Here, the comma denotes partial differentiation with respect to the corresponding coordinates.

The stress tensor \( \bar{\sigma}_{kl} \) of the blood is given by

\[ \bar{\sigma}_{kl} = -\bar{p} g_{kl} \]  

(2)

where \( g_{kl} \) is the metric tensor of the spatial coordinate frame.

2.2. Equations of solid body. The arterial wall material is known to be incompressible anisotropic and viscoelastic (Fung et al., 1979; Cox, 1975; Vaishnav et al., 1972). For the sake of simplicity in the analysis, in this work we shall assume that the arterial wall is incompressible, homogeneous, elastic and isotropic. The initial stress field under the effect of the mean pressure \( P_1 \) is given in cylindrical coordinates by (Demiray, 1976)

\[ t_{rr} = \frac{\beta}{\lambda^2} \int_{x_0}^{x_1} \left( \xi + \frac{\lambda}{\kappa} \right) F(\xi) \, d\xi, \quad t_{\theta\theta} = t_{rr} + \beta \left( \frac{1}{\lambda^2} - \frac{x^2}{\lambda^2} \right) F(x), \quad x = \frac{R}{r} \]

\[ t_{zz} = t_{rr} + \beta \left( \lambda^2 - \frac{x^2}{\lambda^2} \right) F(x), \quad P^\alpha - t_{rr} = \beta \frac{x^2}{\lambda^2} F(x) \]

\[ P_1 = \frac{\beta}{\lambda^2} \int_{x_0}^{x_1} \left( \xi + \frac{\lambda}{\kappa} \right) F(\xi) \, d\xi; \quad F(\xi) = \exp \left[ \alpha \left( \frac{\xi^2}{\lambda^2} + \frac{1}{\xi^2} + \lambda^2 - 3 \right) \right] \]  

(3)

and zero for other components. In obtaining these stress components we have made use of the stress–strain relations of the form

\[ t_{kl} = P^\alpha g_{kl} + \beta \exp[\alpha(I_1 - 3)] c_{kl}; \quad W = \frac{\beta}{2\alpha} \exp[\alpha(I_1 - 3)]. \]  

(4)

In these equations \( t_{kl} \) is the stress tensor, \( P^\alpha \) the hydrostatic pressure, \( W \) the strain energy density, \( c_{kl} \) the Finger deformation tensor, \( I_1 = c_{11} + c_{22} + c_{33} \) the first invariant of \( c_{kl} \), \( \lambda \) the stretch ratio in the axial direction, \( R \) and \( r \) the undeformed and deformed radial coordinates of a material point, \( \alpha, \beta \) two material constants and subscripts \( (i) \) and \( (o) \) are used to denote the evaluations on the inner and outer surfaces of the artery, respectively.

Upon application of the pulse pressure by the left ventricle, a dynamical displacement and stress fields will be superimposed on the above described initial static deformation. Considering the mean pressure for a normal person is
about 100 mm Hg and the pulse pressure is 40( ± 20) mm Hg, the superposed dynamical deformation might remain small as compared to large initial static deformations. Throughout this work, the superposed dynamical deformations will be assumed to be small.

The derivation of the governing field equations and the constitutive relations for small deformation superimposed on a given large static deformation had been studied by Green and Zerna (1968) and Eringen and Suhubi (1974). The governing field equations may be given by

$$\tau_{kl,t} = \rho u_{k,t}$$

(5)

where \( \rho \) is the mass density of solid body, \( u \) is the incremental displacement vector and the incremental stress tensor \( \tau_{kl} \) is defined by

$$\tau_{kl} = \sigma_{kl} + s_{kl}$$

(6)

with

$$s_{kl} = u_{k,m} t_{ml}, \quad \sigma_{kl} = p \delta_{kl} - 2 \rho^o e_{kl} + \Phi c_{kl}$$

$$\Phi = 2 \alpha \beta \exp[\alpha(I_1 - 3)] c_{km} e_{mk}; \quad e_{kl} = \frac{1}{2}(u_{k,t} + u_{l,t}).$$

(7)

Here the summation convention applies to repeated indices and the indices following a semi-colon are used for covariant differentiation.

The incompressibility condition imposes a further restriction on the displacement field \( u \), i.e.

$$u_{k,k} = 0.$$  

(8)

In order to determine the incremental mechanical field completely, equations (5)–(8) are to be supplemented with the boundary conditions which read

$$\sigma_{kl} n_k = \sigma_l \quad \text{(or, } u \text{ is specified) on } S$$

(9)

where \( n_k \) is the unit exterior normal and \( \sigma_l \) is the surface traction.

For the particular problem that we will study here, we shall consider the axially symmetric motion of such a prestressed circular cylindrical shell. For this purpose we set

$$u_1 = u(r, z, t), \quad u_2 = v = 0, \quad u_3 = w(r, z, t)$$

(10)

where \( u \) and \( w \) are the incremental displacement components in the radial and axial directions, respectively. Substituting equations (10) into equations (6) and (7) the physical components of the incremental stress tensor are given by

$$\sigma_{rr} = p + 2\left(\alpha_1 u_r + \alpha_2 \frac{u}{r} + \alpha_3 w_{,z}\right)$$

where \( \alpha_i \) are constants determined by the material properties of the shell.
\[ \sigma_{\theta\theta} = p + 2 \left( \alpha_2 u_r + \alpha_4 \frac{u}{r} + \alpha_5 w_z \right) \]
\[ \sigma_{zz} = p + 2 \left( \alpha_3 u_r + \alpha_5 \frac{u}{r} + \alpha_6 w_z \right), \quad \sigma_{rz} = -P^o (u_z + w_r) \]
\[ s_{rr} = t_{rr} u_r, \quad s_{\theta\theta} = t_{\theta\theta} \frac{u}{r}, \quad s_{zz} = t_{zz} w_z \]
\[ s_{rz} = t_{rz} w_z, \quad s_{sr} = t_{sr} w_r \] (11)

where the coefficients \( \alpha_i \) \((i = 1, 2, \ldots, 6)\) are defined by
\[ \alpha_1 \equiv \alpha \beta (x/\lambda)^4 F(x) - P^o; \quad \alpha_2 \equiv \alpha \beta F(x)/\lambda^2; \quad \alpha_3 \equiv \alpha \beta x^2 F(x) \]
\[ \alpha_4 \equiv \alpha \beta F(x)/x^4 - P^o; \quad \alpha_5 \equiv \alpha \beta (x/\lambda)^2 F(x); \quad \alpha_6 \equiv \alpha \beta \lambda^4 F(x) - P^o. \] (12)

The governing field equations in cylindrical polar coordinates are given by
\[ \tau_{rr,r} + \tau_{rz,z} + \frac{1}{r} (\tau_{rr} - \tau_{\theta\theta}) = \rho u_{,r} \]
\[ \tau_{rz,r} + \tau_{zz,z} + \frac{1}{r} \tau_{rz} = \rho w_{,r}. \] (13)

Introducing equation (11) into equation (13), together with the incompressibility condition, we obtain
\[ p_{,r} + \beta_1(r) u_{,rr} + \beta_2(r) u_{,r} / r - \beta_3(r) u_{,r} / r^2 + \beta_4(r) u_{,zz} = \rho u_{,r} \]
\[ p_{,z} + \beta_5(r) w_{,rr} + \beta_6(r) w_{,r} / r + \beta_7(r) w_{,zz} + \beta_8(r) u_{,z} / r = \rho w_{,r} \]
\[ u_{,r} + u / r + w_{,z} = 0 \) (incompressibility) \] (14)

where the coefficients \( \beta_i(r) \) \((i = 1, 2, \ldots, 8)\) are defined by
\[ \beta_1(r) \equiv t_{rr} + P^o + 2(\alpha_1 - \alpha_3); \quad \beta_2(r) \equiv 2 \frac{d}{dr} \left[r(\alpha_1 - \alpha_3)\right] + 2(\alpha_5 - \alpha_3) + t_{\theta\theta} + P^o \]
\[ \beta_3(r) \equiv t_{\theta\theta} + P^o + 2(\alpha_4 - \alpha_5) + 2r \frac{d}{dr} (\alpha_3 - \alpha_2); \quad \beta_4(r) \equiv t_{zz} - P^o \]
\[ \beta_5(r) \equiv t_{rr} - P^o, \quad \beta_6(r) \equiv t_{\theta\theta} - \frac{d}{dr} (r P^o) \]
\[ \beta_7(r) \equiv 2(\alpha_6 - \alpha_3) + P^o + t_{zz}; \quad \beta_8(r) \equiv -r \frac{d P^o}{dr} + 2(\alpha_2 - \alpha_3). \] (15)
These differential equations are supplemented by the boundary conditions given by
\[ \sigma_{rr}(r_o) = \sigma_{zz}(r_o) = \sigma_{rz}(r_o) = 0 \]
\[ u_z(r_0) = -\frac{u}{\rho} = 0. \] (16)

Here the effect of tethering on the conditions at the outer boundary has been neglected. As might be seen from equations (1), a closed form of solution may be given for equations governing the fluid body. However, due to variability of the coefficients, a closed form solution to the differential equations governing the incremental motion of the solid cannot be obtained. We shall, therefore, present a method which utilizes the power series expansion of the field quantities.

Thus far the researchers working in this area [cf. Atabek (1968), Rachcoff (1980) and Demiray (1985)] have treated the arteries as thin-walled cylindrical shells and used the membrane equations for the analysis. As is well known, however, thin shell theories are valid when the ratio of thickness to mean radius is less than 0.05. But, for most arteries, this ratio varies between 1/4 and 1/6 which is quite large as compared to allowable limits for membrane theories. This consideration clearly indicates that the membrane theory cannot be applied to arteries which appear to be a thick cylindrical shell.

3. Solution to Field Equations. In this section we shall seek a harmonic wave type of solution to field equations (1) and (14). The appropriate form of the concerned variables should be as follows:

\[ (\tilde{u}, \tilde{w}, \tilde{p}) = [\tilde{U}(r), \tilde{W}(r), \tilde{P}(r)] \exp(i\omega t - k z) \] (17a)

\[ (u, w, p) = [\tilde{U}(r), \tilde{W}(r), \tilde{P}(r)] \exp(i\omega t - k z) \] (17b)

where \( \omega \) is the angular frequency, \( k \) the wave number and \( \tilde{U}(r), \ldots, \tilde{P}(r) \) unknown functions characterizing the wave amplitudes.

The solutions \( \tilde{U}(r), \tilde{W}(r), \tilde{P}(r) \) satisfying equations (1) may be given by

\[ \tilde{U}(r) = A k I_1 (kr); \quad \tilde{W}(r) = -i A k I_0 (kr), \quad \tilde{P} = -i \omega A \tilde{I}_0 (kr) \] (18)

where \( A \) is an integration constant to be determined from the boundary conditions and \( I_1 (kr) \) is the modified Bessel function of the first kind.

In order to obtain the solution for the elastic solid body we first introduce equation (17b) into equations (14) obtaining the following set of ordinary differential equations

\[ \tilde{P}_s + \beta_1 (r) \tilde{U}_s + \beta_2 (r) \tilde{W}_s/r - \beta_3 (r) \tilde{U}/r^2 + (\rho \omega^2 - k^2 \beta_4 (r)) \tilde{U} = 0 \]

\[ -ik \tilde{p} + \beta_5 (r) \tilde{W}_s + \beta_6 (r) \tilde{W}_s/r + (\rho \omega^2 - k^2 \beta_7 (r)) \tilde{W} - ik \beta_8 (r) \tilde{U}/r = 0 \]

\[ \tilde{U}_s + \tilde{U}/r - ik \tilde{W} = 0. \] (19)
At this point it might be useful to introduce the following dimensionless quantities

\[
\bar{\beta}_i(r) = \beta \beta_i(r) \quad (i = 1, 2, \ldots, 8), \quad \bar{P} = \beta P \\
\bar{U} = \bar{r} U, \quad \bar{W} = \bar{r} W, \quad r = \bar{r}(1 + y); \quad \left( -\frac{h}{2\bar{r}} \leq y \leq \frac{h}{2\bar{r}} \right) \\
c_0^2 = \beta / \rho, \quad \omega = \frac{c_0}{\bar{r}} \Omega, \quad \xi = k\bar{r}
\]

where \( \bar{r} \) is the deformed mean radius and \( h \) is the deformed thickness of the arterial wall. If expressions (20) are introduced into equations (19) the following set of ordinary differential equations are obtained

\[
P_{,y} + \beta_1(y) U_{,xy} + \beta_2(y) U_{,y} + \beta_3(y) U/(1 + y) - \beta_3(y) U/(1 + y)^2 + (\Omega^2 - \xi^2 \beta_4(y)) U = 0 \\
- i\xi P + \beta_5(y) W_{,yy} + \beta_6(y) W_{,y} + (\Omega^2 - \xi^2 \beta_7(y)) W - i\xi \beta_8(y) U/(1 + y) = 0 \\
U_{,y} + U/(1 + y) - i\xi W = 0.
\]

(21)

Here the explicit expressions of \( \beta_i(y) \), with the aid of equations (3) and (12), are given by

\[
\beta_1(y) = \left[ \frac{x^2}{\lambda^2} + 2\alpha \left( \frac{x^4}{\lambda^4} - x^2 \right) \right] F(x) \\
\beta_2(y) = \left[ 2\alpha \left( \frac{x^4}{\lambda^4} + \frac{\lambda^2}{x^2} - 2x^2 \right) + \frac{1}{x^2} \right] F(x) + \frac{2}{\beta} (1 + y) \frac{d}{dy} (\alpha_1 - \alpha_3) \\
\beta_3(y) = \left[ 2\alpha \left( \frac{1}{x^4} - \frac{\lambda^2}{x^2} \right) + \frac{1}{x^2} \right] F(x) + \frac{2}{\beta} (1 + y) \frac{d}{dy} (\alpha_3 - \alpha_2) \\
\beta_4(y) = \lambda^2 F(x), \quad \beta_5(y) = \frac{x^2}{\lambda^2} F(x), \quad \beta_6(y) = \frac{1}{x^2} F(x) - \frac{(1 + y)}{\beta} \frac{dP^o}{dy} \\
\beta_7(y) = [2\alpha(\lambda^4 - x^2) + \lambda^2] F(x), \quad \beta_8(y) = - \frac{(1 + y)}{\beta} \frac{dP^o}{dy} + \frac{2}{\beta} (\alpha_5 - \alpha_3).
\]

(22)

In this case, dropping the exponential factor, the components of the stress tensor that we shall need in the boundary conditions take the following form

\[
\sigma_{\bar{r}y}(y) = \beta P + (\alpha_1 \frac{dU}{dy} + \alpha_2 U/(1 + y) - i\xi \alpha_3 W) \\
\sigma_{\bar{r}z}(y) = - P^o \left( \frac{dW}{dy} - i\xi U \right); \quad -\frac{h}{2\bar{r}} \leq y \leq \frac{h}{2\bar{r}}.
\]

(23)
The complex structures of the coefficient functions in equations (21) make it impossible to obtain a closed form analytic solution to the field equations of the solid continuum. We shall, therefore, present a power series approximation to the field equations. For this purpose, we set

$$
\beta_i(y) = \beta_i^{(0)} + \beta_i^{(1)}y + \beta_i^{(2)}y^2 + \ldots \quad (i = 1, 2, \ldots, 8)
$$

$$
\alpha_i(y) = \beta(\alpha_i^{(0)} + \alpha_i^{(1)}y + \alpha_i^{(2)}y^2 + \ldots) \quad (i = 1, 2, \ldots, 6)
$$

$$
U(y) = U_0 + U_1y + U_2y^2 + \ldots
$$

$$
W(y) = W_0 + W_1y + W_2y^2 + \ldots
$$

$$
P(y) = P_0 + P_1y + P_2y^2 + \ldots
$$

$$
P^0(y) = P^0_0 + P^0_1y + P^0_2y^2 + \ldots
$$

(24)

where the coefficients $\beta_i^{(0)}, \ldots, \alpha_i^{(2)}, \ldots, P^0_2$ are constants to be determined from the series expansion of the corresponding known functions while $U_0, \ldots, P_2$ are constants to be determined from the field equations and the boundary conditions. If equations (24) are introduced into equations (21) and the coefficients of the same power of $y$ are set equal to zero we obtain an infinite set of algebraic equations relating some of the coefficients to the others. It is, of course, impossible to get any tractable result by keeping all the terms in the expansion. For our future purposes, however, we shall only retain the terms up to and including $y^2$. Then, when combined with equations (24), equations (21) reduce to

$$
P_1 + (\Omega^2 - \beta_3^{(0)} - \beta_4^{(0)}\xi^2)U_0 + \beta_2^{(0)}U_1 + 2\beta_1^{(0)}U_2 = 0
$$

$$
-i\xi P_0 - i\xi \beta_8^{(0)}U_0 + (\Omega^2 - \beta_7^{(0)}\xi^2)W_0 + \beta_6^{(0)}W_1 + 2\beta_5^{(0)}W_2 = 0
$$

$$
U_0 + U_1 - i\xi W_0 = 0
$$

$$
-U_0 + U_1 + 2U_2 - i\xi W_1 = 0.
$$

(25)

To be consistent with equations (25) the incremental stress components that we shall use in the boundary conditions must be linear in the variable $y$. Hence, from equations (23) we have

$$
\sigma_{rr}/\beta = P_0 + 2(\alpha_1^{(0)}U_1 + \alpha_2^{(0)}U_0 - i\xi \alpha_3^{(0)}W_0) + y\{\beta P_1 + 2[(\alpha_2^{(1)} - \alpha_2^{(0)})U_0
$$

$$
+ (\alpha_1^{(1)} + \alpha_2^{(0)})U_1 + 2\alpha_1^{(0)}U_2 - i\xi (\alpha_3^{(1)}W_0 + \alpha_3^{(0)}W_1)]\} + O(y^2)
$$

$$
\sigma_{rz} = -P^0_0(W_1 - i\xi U_0) - [P^0_0(2W_2 - i\xi U_1) + P^0_0(W_1 - i\xi U_0)]y + O(y^2).
$$

(26)

The number of unknown constants appearing in the field equations (18) and (25) is nine whereas the number of boundary conditions is five. Considering that equations (25) give four additional relations among the unknown
constants we have enough equations to determine all the integration constants. In the first place, by use of equations (25) one can eliminate \( P_1, U_1, U_2 \) and \( W_2 \) from the expression of stresses, equations (26). Thus we have

\[
\begin{align*}
\sigma_{rr} &= \beta \{ P_o + 2(\gamma_1 U_0 + i\xi\gamma_2 W_0) + \gamma[(\gamma_3 \xi^2 - \Omega^2 - \gamma_4)U_0 + i\xi \gamma_5 W_0] \nonumber \\
&- i\xi \gamma_6 W_1 \} \} + O(y^2) \\
\sigma_{rz} &= - P_o (W_1 - i\xi U_0) - y \frac{P_o}{\beta S_o} [i\xi P_o + i\xi \gamma U_0 + (\gamma_8 \xi^2 - \Omega^2)W_0] \nonumber \\
&+ \gamma_9 W_1 \} + O(y^2)
\end{align*}
\]

where the coefficients \( \gamma_i (i = 1, 2, \ldots, 9) \) are defined by

\[
\begin{align*}
\gamma_1 &= \left( \frac{\alpha}{\lambda^2} - \frac{\alpha \delta^4}{\lambda^4} - \frac{\delta^2}{\lambda^2} \right)F_0 - \Gamma_0, \\
\gamma_2 &= \left( \frac{\alpha \delta^4}{\lambda^4} + \frac{\delta^2}{\lambda^2} - \alpha \delta^2 \right)F_0 + \Gamma_0 \\
\gamma_3 &= \lambda^2 F_0, \\
\gamma_4 &= -2 \left[ \left( \frac{\delta^2}{\lambda^2} + \frac{1}{\delta^2} \right) + \alpha \left( \frac{\delta^2}{\lambda^2} - \frac{1}{\delta^2} \right)^2 \right] F_0 - 4 \Gamma_0 \\
\gamma_5 &= - \left[ \left( \frac{1}{\delta^2} + \frac{2 \alpha \lambda^4}{\lambda^2 - \delta^2} \right) + 2 \alpha \left( \frac{\delta^2}{\lambda^4} + \frac{\lambda^2}{\delta^2 - \delta^2} - \frac{1}{\lambda^2} \right) \right] F_0 - 2 \Gamma_0 \\
\gamma_6 &= - \frac{\delta^2}{\lambda^2} F_0 - 2 \Gamma_0 \\
\gamma_7 &= F_0 \left\{ \frac{\delta^2}{\lambda^2} \left[ 1 + \Gamma_1 \left( \Gamma_0 + \frac{\delta^2}{\lambda^2} F_0 \right)^{-1} \right] + 2 \alpha \left( \frac{\lambda^2}{\delta^2} - \delta^2 \right) \right\} - \Gamma_1 \\
\gamma_8 &= \left[ \lambda^2 + \frac{\delta^2}{\lambda^2} + 2 \alpha (\lambda^4 - \delta^2) \right] F_0 \\
\gamma_9 &= - F_0 \left[ \frac{1}{\delta^2} + \Gamma_1 \frac{\delta^2}{\lambda^2} \left( \Gamma_0 + \frac{\delta^2}{\lambda^2} F_0 \right)^{-1} \right] + \Gamma_1
\end{align*}
\]

with

\[
\begin{align*}
\Gamma_0 &= \frac{1}{\lambda^2} \int_\delta^{\infty} \left( v + \frac{\lambda}{v} \right) F(v) \, dv, \quad F_0 = \exp \left[ \alpha \left( \frac{\delta^2}{\lambda^2} + \frac{1}{\delta^2} + \lambda^2 - 3 \right) \right] \\
\Gamma_1 &= \left( \frac{\delta^2 - \lambda}{\lambda^2} \right) \left[ \left( 1 - \frac{\lambda}{\delta^2} \right) + 2 \alpha \left( \frac{\delta^2}{\lambda^2} - \frac{1}{\delta^2} \right) \right] F_0
\end{align*}
\]
\[ P_0^o = -\beta \left( \frac{\delta^2}{\lambda^2} F_0 + \Gamma_0 \right), \quad P_1^o = \beta \Gamma_1 \]
\[ \delta^2 = \left( \bar{x} - \frac{\lambda}{\bar{x}} \frac{h}{2\bar{r}} \right)^2 - \lambda \left( 1 - \frac{h}{2\bar{r}} \right)^2 + \lambda; \quad \bar{x} = \frac{R}{\bar{r}}. \] (29)

Here \( R \) and \( \bar{r} \) are the midradius of the cylindrical artery before and after deformation, respectively. In obtaining these expressions we expanded various quantities into a power series of \( y \) and kept only up to the linear terms, e.g.
\[ x \approx \delta - \left( \frac{\delta^2 - \lambda}{\delta} \right)y + O(y^2). \] (30)

If the initial deformation vanishes, i.e. \( \bar{x} = 1 \) and \( \lambda = 1 \), from equations (28) and (29) we have \( \delta = 1, \quad -\gamma_1 = \gamma_2 = \gamma_3 = -\gamma_6 = -\gamma_7 = -\gamma_9 = 1, \quad \gamma_4 = -4 \) and \( \gamma_5 = -\gamma_8 = -2 \). This special case will be discussed later.

We shall now try to satisfy the boundary conditions given in equations (16). If equations (18), (24) and (27) are introduced in equations (16) one obtains
\[ q[P_0 + 2(\gamma_1 U_0 + i\xi \gamma_2 W_0)] - m[(\gamma_3 \xi^2 - \Omega^2 - \gamma_4) U_0 + i\xi \gamma_5 W_0 - i\xi \gamma_6 W_1] \]
\[ -i\Omega AI_0 \left[ \xi \left( 1 - \frac{h}{2\bar{r}} \right) \right] = 0 \]
\[ q[P_0 + 2(\gamma_1 U_0 + i\xi \gamma_2 W_0)] + m[(\gamma_3 \xi^2 - \Omega^2 - \gamma_4) U_0 + i\xi \gamma_5 W_0 - i\xi \gamma_6 W_1] = 0 \]
\[ W_1 - i\xi U_0 = 0 \]
\[ i\xi P_0 + i\xi \gamma_7 U_0 + (\gamma_8 \xi^2 - \Omega^2) W_0 + \gamma_9 W_1 = 0 \]
\[ i\Omega \left( 1 + \frac{h}{2\bar{r}} + \frac{h^2}{4\bar{r}^2} \right) U_0 + \xi\Omega \left( \frac{h}{2\bar{r}} + \frac{h^2}{8\bar{r}^2} \right) W_0 - \xi\Omega \left( \frac{h^2}{8\bar{r}^2} \right) W_1 - \xi I_1 \left[ \xi \left( 1 - \frac{h}{2\bar{r}} \right) \right] A = 0 \] (31)

where, some new parameters appearing in the above equations are defined by
\[ q = \rho/\bar{\rho}, \quad m = qh/2\bar{r}, \quad A = \bar{r}c_0 A. \] (32)

In order to have a non-zero solution for the coefficients \( A, P_0, U_0, W_0 \) and \( W_1 \), the determinant of the coefficient matrix obtained from the set of equations (31) must vanish. If this is done, after a lengthy manipulation we obtain the following dispersion relation
\[ v^4 \left\{ 1 + \frac{h}{2\bar{r}} + \frac{h^2}{4\bar{r}^2} + \xi^2 \left[ mF \left( \xi \left( 1 - \frac{h}{2\bar{r}} \right) \right) \left( 1 - \frac{h}{2\bar{r}} \right) + \frac{h^2}{8\bar{r}^2} + \frac{h^3}{16\bar{r}^3} \right] \right\} \]
\[-v^2 \left\{ K_1 + mK_3F\left( \xi \left( 1 - \frac{h}{2F} \right) \right) \left( 1 - \frac{h}{2F} \right) + \frac{h}{2F} K_2 + \frac{h^2}{8F^2} + (K_1 + K_2 + 2K_3 + K_4) \right. \\
\left. + \frac{h^3}{16F^3} (K_3 + 2K_4) + \xi^2 \left[ mF\left( \zeta \left( 1 - \frac{h}{2F} \right) \right) \left( 1 - \frac{h}{2F} \right) (K_1 + K_3) \right. \\
\left. + \frac{h^3}{16F^3} (K_3 - K_4) \right] \right\} + mF\left( \xi \left( 1 - \frac{h}{2F} \right) \right) \left( 1 - \frac{h}{2F} \right) (K_6 + \xi^2 K_1 K_5) = 0 \tag{33}\]

where \( v = \Omega / \zeta = c / c_0 \) (\( c \) is the phase velocity) and other quantities are defined by

\[
F(z) = 2I_1(z)/zI_0(z), \quad K_1 = 2\gamma_2 + \gamma_8, \\
K_2 = \gamma_8 - (\gamma_4 + \gamma_9) + \gamma_5 + 2(\gamma_1 + \gamma_2), \\
K_3 = -\gamma_4, \quad K_4 = \gamma_9, \quad K_5 = \gamma_3 + \gamma_6, \\
K_6 = \gamma_5(\gamma_7 + \gamma_9) - \gamma_4\gamma_8 - 2\gamma_1\gamma_5 - 2\gamma_2\gamma_4. \tag{34}\]

Equation (33) is the most general dispersion relation obtained in this approximate solution. As is seen from equation (33) the phase velocity \( v \) depends on the wave number \( \xi \), that is the wave is dispersive. Before discussing the more general case it might be instructive to investigate some special cases.

3.1. Thin tube with initial stress. The dispersion equation of an initially stressed thin tube may be obtained from equation (33) by setting \( h/F \to 0 \) and keeping the parameter \( m \) finite [see, Rubino and Keller (1971)]. This assumption about \( m \) is required in order to account for the effect of the inertia of the solid tube. In this case, noting that \( \delta^2 = \xi^2 \), from equation (33) we have

\[
v^4\left[ 1 + \xi^2 mF(\xi) \right] - v^2 \left[ K_1 + mK_3F(\xi) + \xi^2 m(K_1 + K_3)F(\xi) \right] + mF(\xi) (K_6 + \xi^2 K_1 K_5) = 0. \tag{35}\]

When the initial deformation vanishes, i.e. \( \tilde{\xi} = \lambda = 1 \), then \( K_1 = K_2 = 4, \ K_4 = -2, \ K_6 = 12 \ and \ K_2 = K_3 = 0 \). Considering that \( K_i = K_i(\tilde{\xi}, \lambda) \) (\( i = 1, 2, \ldots, 6 \)), equation (35) shows that in general the phase velocity depends on the initial deformations (or the inner pressure and an axial force).

For the case of vanishing initial deformation the dispersion relation reduces to

\[
v^4\left[ 1 + \xi^2 mF(\xi) \right] - 4v^2 \left[ 1 + m(1 + \xi^2)F(\xi) \right] + 12mF(\xi) = 0. \tag{36}\]
This is exactly the same as that obtained by Rubinow and Keller (1971) for thin elastic tubes provided that one sets \( v = 1/2 \) and \( E = 3\beta \) there. This equation still indicates that the phase velocity is dependent upon the wave number, that is, the wave is dispersive.

If the wavelength is very large as compared to the midradius (or inner radius for thin tubes), i.e. \( \xi \to 0 \) then \( F(\xi) \to 1 \) and from equation (35) the dispersion relation for a thin tube becomes

\[
v^4 - v^2(K_1 + mK_3) + mK_6 = 0. \tag{37}
\]

The solution of this quadratic equation yields

\[
v_{1,2}^2 = \frac{1}{2} \left( K_1 + mK_3 \mp \sqrt{(K_1 + mK_3)^2 - 4mK_6} \right). \tag{38}
\]

For thin tubes the parameter \( m \) is a small quantity (Rubinow and Keller, 1971). In this case one can approximate the phase velocities as

\[
v_1^2 = K_1 + O(m), \quad v_2^2 = \frac{K_6}{K_1} m + O(m^2). \tag{39}
\]

In order to see what these terms mean, we shall first set the initial deformation equal to zero

\[
v_1^2 = 4, \quad v_2^2 = 3m
\]

or, in terms of real dimensional physical quantities

\[
c_1^2 = \frac{4\beta}{\rho} \quad \text{and} \quad c_2^2 = \frac{3\beta}{\rho} \frac{h}{2r}. \tag{40}
\]

If we set, as before, \( E = 3\beta \), the phase velocities become

\[
c_1^2 = \frac{4}{3} \frac{E}{\rho}, \quad c_2^2 = \frac{E}{\rho} \frac{h}{2r}. \tag{41}
\]

Of these wave speeds \( c_1 \) corresponds to the Lamb mode while \( c_2 \) to the Moens–Korteweg (Young’s mode) formula for the pulse wave speed in arteries. The last result is also consistent with the findings obtained from the method of characteristics (Demiray, 1985). Having included the effect of thickness and the large initial deformation, the present work may be considered as the generalization of the previous works on the same subject.

Employing the experimental results of Simon et al. (1972) on canine abdominal arteries, whose characteristics are \( R_i = 3.12 \times 10^{-3} \text{ m}; R_o = 3.8 \times 10^{-3} \text{ m} \), the values of material constants were found to be \( \alpha = 1.948 \) and
\( \beta = 9900 \text{ Pa} \) for \( \lambda = 1.53 \) (Demiray, 1976). Utilizing these material and geometrical characteristics in the general dispersion relations various quantities are numerically evaluated and the results are depicted in Figs 1–5. Figure 1 shows the variations of the speed of first mode (Lamb mode) with wave number, axial stretch and the inner pressure. As is seen from the graph the phase velocity of the primary wave is almost constant in wave number whereas it increases with increasing stretch and transmural pressure. The variation of the phase velocity of the secondary wave (Young's mode), which is depicted in Fig. 2, reveals that the wave speed increases with wave number and the initial deformation. It should be noticed that, when the initial deformation vanishes the speed decreases with increasing wave number. Figures 3 and 4 show the variations of the corresponding frequencies with non-dimensional wave

![Diagram](image-url)

Figure 1. Variation of phase velocity (Lamb's mode) with wave number.
Figure 2. Variation of phase velocity (Young's mode) with wave number.

Figure 3. Variation of frequency (Lamb's mode) with wave number.
number. As is seen, the relation in Fig. 3 is almost linear, while it is highly curvilinear in Fig. 4. Finally, the variation of phase speeds with inner pressure for various stretch ratios is given in Fig. 5. It reveals that the speeds of both modes increase with inner pressure and the axial stretch. Furthermore, as the inner pressure increases, beyond a certain limit the pulse speeds become less sensitive to the increments in the pressure. This means that from that point on the incremental stiffness of the arterial wall does not change with pressure. This behaviour is peculiar to soft biological tissues. Although it has not been shown on these figures, the thin tube theories estimate various wave characteristics larger (∼5%) than the thick shell theories.
Figure 5. Variations of phase velocities with inner pressure.

LITERATURE


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