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Effect of Prestress on Pulse Waves in Arteries


In order to understand the effect of initial stresses on the blood flow in arteries, a theoretical analysis of wave propagation in an initially inflated and axially stretched thick cylindrical shell filled with a viscous fluid is investigated. For simplicity in mathematical analysis, the blood is assumed to be an incompressible Newtonian fluid while the arterial wall material is taken to be incompressible, homogeneous, isotropic and elastic. Utilizing the theory of small deformations superimposed on a large initial deformation, the governing differential equations of the perturbed solid motions are obtained in cylindrical polar coordinates. Because of the difficulties in solving the set of differential equations with variable coefficients, an approximate power series method is utilized and the dispersion relation is obtained for thin tubes. Various results concerning the wave speeds and transmission coefficients are numerically evaluated and the results are depicted on some graphics.

Для выяснения влияния начального напряжения на течение крови в артериях проводится теоретический анализ распространения волн в начально напряженной, вытянутой в осевом направлении толстой цилиндрической оболочке, заполненной вязкой жидкостью. Для упрощения математического анализа в качестве напряженного состояния принимается постоянный объем, однородный, изотропный и упругий. Используя теорию малых деформаций, наложенных на большую начальную деформацию, дифференциальные уравнения возмущенного движения твёрдого тела описываются в цилиндрических координатах. Ввиду трудностей при решении системы дифференциальных уравнений с переменными коэффициентами, использован приближенный метод степенных рядов, в результате чего получено дисперсионное соотношение для тонких труб. Различные результаты, касающиеся скоростей волн и коэффициентов пропускания, обработаны численно и изображены графически.

1. Introduction

Propagation of harmonic waves in an initially stressed (or unstressed) cylindrical tube filled with viscous (or inviscid) fluid is a problem of interest since the time of Thomas Young who first (to our knowledge) studied the pulse wave speed in human arteries. This subject, in particular, has received considerable interest and activity among the research workers in the last two or three decades. The current literature on the subject is so rich that one cannot mention all the contributed works here. The historical evolution of the problem may be found in the paper by Lambossy [1] and Skalak [2] and in the books by Attinger [3], McDonald [4] and Fung [5]. Significant contribu-

tions on wave motions of an elastic tube filled with a viscous fluid have been given by Morgan and Keely [6], Womersley [7], and Atabek and Lew [8]. All these researchers have assumed that the arterial wall is a thin walled isotropic elastic tube. As pointed out in their experimental studies (Lawton [9] and Fern [10]), the artery may have thick walls with viscoelastic and anisotropic properties. These characteristics of blood vessels have partially been taken into account by Minsky [11], Atabek [12], Rubink and Keller [13] and more recently by Rachey [14] and Kuiken [15]. However, in all these works, either the effects of initial stresses have been neglected or the artery has been treated as a membrane. In reality, the arteries are thick walled and subjected to large initial stresses.

In the present work, making use of the field equations and boundary conditions of a theory known as "small deformations superimposed on large initial deformations", the propagation of harmonic waves in a cylindrical elastic tube filled with an incompressible viscous fluid is investigated. For simplicity in the analysis, the arterial wall is taken to be an isotropic and incompressible elastic material and then the governing differential equations of the solid body are obtained in cylindrical polar coordinates. Due to variability of the initial stresses through the thickness, the coefficients of the resulting differential equations are also variable. Considering the difficulties in dealing with such a set of differential equations, a truncated power series method is utilized. Employing the boundary conditions the dispersion relation is obtained as a function of inner pressure, axial stretch and the thickness ratio. Following Rubink and Keller [13], in order to have a tangible result, the dispersion relation is obtained for thin tubes with or without initial stresses. In the absence of viscosity and for wavelengths large as compared to mean radius, the wave speeds reduce to Lamé and Moens-Korteweg (Young) modes. The numerical results further indicate that the wave speeds increase with increasing inner pressure and axial stretch. The transmission coefficient of the dominating wave decreases with increasing pressure while it increases with increasing axial stretch. Conversely, the transmission coefficient of the secondary wave increases with pressure and decreases with axial stretch.

2. Theoretical preliminaries

Due to interactions of blood with its container, the pulsatile motion of blood leads to wave phenomena in arteries. The governing field equations and the boundary conditions should, therefore, include these interactions.

Fluid Equations: The blood is known to be an incompressible non-Newtonian fluid. However, for the sake of its simplicity in the analysis we shall treat the blood as an incompressible, viscous fluid which is subjected to a constant mean pressure $P$. When such an equilibrium state of the blood is disturbed by a pressure pulse $\bar{p}(r, z, t)$ generated by the left ventricle, a harmonic wave type of flow field will be developed in the blood. Assuming the initial average velocity field vanishes, for axially symmetric situations in the cylindrical polar coordinates are given by

$$\begin{align*}
- \frac{\partial p}{\partial r} + \mu \left( \frac{\partial u}{\partial r} + 1 \frac{\partial \bar{u}}{r \partial r} - \frac{\bar{u}}{r^2} + \frac{\partial u}{\partial z} \right) &= \frac{\partial \bar{u}}{\partial t}, \\
- \frac{\partial p}{\partial z} + \mu \left( \frac{\partial \bar{w}}{\partial r} + 1 \frac{\partial \bar{w}}{r \partial r} + \frac{\partial \bar{u}}{\partial z} \right) &= \frac{\partial \bar{w}}{\partial t}, \\
\frac{\partial \bar{u}}{\partial r} + \bar{u} &= \frac{\partial \bar{w}}{\partial z} = 0
\end{align*}$$

(1)

where $\bar{u}$ is the fluid mass density, $\mu$ is the viscosity and $\bar{u}$ and $\bar{w}$ are, respectively, the velocity components of the fluid in the radial and axial directions. The components of stress tensor of the fluid which we need in using the boundary conditions are given by

$$\sigma_{rr} = -p + 2\mu \frac{\partial \bar{u}}{\partial r}, \quad \sigma_{rz} = \mu \left( \frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial r} \right).$$

(2)

Equations of Solid Body: The material of arterial wall is known to be incompressible, anisotropic and viscoelastic (see, Fung et al. [16] and Cox [17]). For its simplicity in nonlinear analysis the arterial wall material shall be assumed to be incompressible, homogeneous, isotropic and elastic. The initial stresses and the deformation fields resulting from the mean pressure $P_1$ and the axial force $N$, in cylindrical polar coordinates, are given by (Demiray [18])

$$\begin{align*}
t_{rr} &= \frac{\xi}{2\alpha} \int_{y_1}^{y_2} \left( \frac{\xi}{\xi} + \frac{\lambda}{\xi} \right) F(\xi) \, d\xi, \\
t_{rs} &= t_{rr} + \beta \left( \frac{1}{\sqrt{\lambda}} - \frac{y^2}{\sqrt{\lambda}} \right) F(y), \\
t_{zz} &= t_{rz} + \beta \left( \frac{y^2}{\sqrt{\lambda}} - \frac{y}{\sqrt{\lambda}} \right) F(y), \\
P_1 &= \frac{\xi}{2\alpha} \int_{y_1}^{y_2} \left( \frac{\xi}{\xi} + \frac{\lambda}{\xi} \right) F(\xi) \, d\xi; \\
F(\xi) &= \exp \left[ \lambda \left( \frac{\xi^2}{\xi^2} + \frac{1}{\xi^2} + 2^2 - 3 \right) \right]
\end{align*}$$

(3)

Here in obtaining these stress components we have made use of the stress-strain relations of the form (Demiray [19])

$$\begin{align*}
t_{st} &= P_{st} y_{tt} + \beta \exp \left[ \lambda (I_1 - 3) \right] c_{st}.
\end{align*}$$

(4)
In this equation, \( \tau_{kl} \) is the stress tensor, \( P^0 \) is the hydrostatic pressure, \( c_{kl} \) is the Finger deformation tensor, \( I_1 = c_{11} + c_{22} + c_{33} \) is the first invariant of \( c_{kl} \), \( \Phi \) is the metric tensor of the spatial frame, \( \lambda \) is the stretch ratio in the axial direction, \( R \) and \( r \) are, respectively, the undeformed and deformed radial coordinates of a material point, \( \alpha \) and \( \beta \) are two material constants and the subscripts \( (i) \) and \( (o) \) are used to denote the evaluations on the inner and outer surfaces of the artery, respectively. For other details of the subject the reader may be referred to DEMRAY [18].

Upon application of the pulse pressure by the left ventricle, an additional dynamical displacement and stress field will be developed in the body. Considering that the mean pressure for a normal person is around 100 mm Hg and the maximum amplitude of the pulse pressure is 20 mm Hg, the superposed dynamical motion might remain small as compared to large initial static deformations.

The derivation of the governing differential equations and the constitutive relations for small deformations superimposed on a given large static deformation had been given by GREEN and ZIRNA [20] and ERingen and SUHUBI [21]. The governing differential equations and the incremental constitutive relations are given by

\[
\tau_{kl} = \frac{\partial^2 u_k}{\partial t^2}
\]

where \( \rho \) is the mass density of the arterial wall, \( u_k \) \((k = 1, 2, 3)\) are the components of the incremental displacement vector and the incremental Piola stress tensor \( \tau_{kl} \) (not symmetric) is defined by

\[
\tau_{kl} = s_{kl} + r_{kl}
\]

with

\[
s_{kl} = u_k w_{ml} \quad \sigma_{kl} = p_{kl} - 2P^0 \delta_{kl} + \Phi_{kl},
\]

\[
\Phi = 2 \lambda \rho \exp \left[ \alpha (I_1 - 3) \right] \psi_{kl} \quad \psi_{kl} = \frac{1}{3} \left( u_{kl} + u_{lk} \right)
\]

Here \( p \) is the increment in hydrostatic pressure, the summation convention applies on repeated indices and the indices following a semi-colon stand for the covariant differentiations.

The field equations (5) — (7) are further restricted by the incompressibility condition, i.e.,

\[
u_{kk} = 0.
\]

In order to determine the incremental mechanical field completely, these field equations are to be supplemented by the boundary conditions which read

\[
\sigma_{kl} w_k = \alpha (or, u_k is specified) on S
\]

where \( n_k \) is the unit exterior normal to \( S \) and \( \alpha \) is the surface traction.

For the special problem that we shall study here, we will consider an axisymmetric motion of such a prestressed circular cylindrical shell. In this case, the physical components of the incremental stress tensor are given by

\[
\sigma_{rr} = p + 2(\alpha_1 \frac{\partial u}{\partial r} + \alpha_2 \frac{u}{r} + \alpha_3 \frac{\partial w}{\partial z}), \quad \sigma_{zz} = p + 2(\alpha_1 \frac{\partial u}{\partial z} + \alpha_4 \frac{u}{r} + \alpha_5 \frac{\partial w}{\partial z}),
\]

\[
\sigma_{zz} = p + 2(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z}), \quad \sigma_{rr} = -\rho_0 \left( \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r} \right),
\]

\[
s_{rr} = \frac{\partial u}{\partial r}, \quad s_{zz} = \frac{\partial w}{\partial z}, \quad s_{rz} = \frac{\partial w}{\partial z}, \quad s_{rr} = \frac{\partial u}{\partial z}, \quad s_{rz} = \frac{\partial u}{\partial z}
\]

where the coefficients \( \alpha_i \) \((i = 1, 2, \ldots, 6)\) are defined by

\[
\alpha_1 \equiv \alpha \beta (y \beta)^3 F(y) - P^0(y); \quad \alpha_2 \equiv \alpha \beta (y \beta)^3 F(y); \quad \alpha_3 \equiv \alpha \beta (y \beta)^3 F(y); \quad \alpha_4 \equiv \alpha \beta (y \beta)^3 F(y); \quad \alpha_5 \equiv \alpha \beta (y \beta)^3 F(y)
\]

and the quantities \( u \) and \( w \) are respectively the incremental displacement components in the radial and the axial directions.

If (10) is introduced into (5) and (6), along with the incompressibility condition, the field equations take the following form

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial r} \frac{\partial^2 u}{\partial r^2} + \frac{\partial \rho}{\partial z} \frac{\partial^2 w}{\partial z^2} = \frac{\partial^2 u}{\partial t^2},
\]

\[
\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial r} \frac{\partial^2 w}{\partial r^2} + \frac{\partial \rho}{\partial z} \frac{\partial^2 w}{\partial z^2} = \frac{\partial^2 w}{\partial t^2},
\]

\[
\frac{\partial u}{\partial t} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \quad \text{(incompressibility)}
\]
where the coefficients $\bar{\beta}_i(r) \ (i = 1, 2, \ldots, 8)$ are defined by

\[
\begin{align*}
\bar{\beta}_1(r) &= \beta \left[ \frac{y^2}{z^2} + 2 \alpha \left( \frac{y^2}{z^2} - y^2 \right) \right] F(y), \\
\bar{\beta}_2(r) &= \beta \left[ 2 \alpha \left( \frac{y^2}{z^2} - y^2 \right) + \frac{1}{y^2} \right] F(y) + 2 \frac{d}{dr} (\alpha_1 - \alpha_2), \\
\bar{\beta}_3(r) &= \beta \left[ 2 \alpha \left( \frac{1}{y^2} - \frac{1}{z^2} \right) + \frac{1}{y^2} \right] F(y) + 2 \frac{d}{dr} (\alpha_2 - \alpha_3), \\
\bar{\beta}_4(r) &= \beta \left[ 2 \alpha \left( \frac{1}{y^2} - \frac{1}{z^2} \right) + \frac{1}{y^2} \right] F(y), \\
\bar{\beta}_5(r) &= \beta \left[ \frac{y^2}{z^2} F(y) \right], \\
\bar{\beta}_6(r) &= \beta \left[ \frac{y^2}{z^2} F(y) \right], \\
\bar{\beta}_7(r) &= \beta \left[ \frac{y^2}{z^2} F(y) \right], \\
\bar{\beta}_8(r) &= \beta \left[ \frac{y^2}{z^2} F(y) \right].
\end{align*}
\]

These differential equations are to be supplemented by the boundary conditions given by

\[
\begin{align*}
\frac{\partial u(r_1, t)}{\partial t} &= \bar{u}(r_1, t), \\
\frac{\partial w(r_1, t)}{\partial t} &= \bar{w}(r_1, t), \\
\end{align*}
\]

Here the effect of tethering on the conditions at the outer boundary has been neglected. As is seen from equations (1), although a closed form solution may be given for fluid body, due to variability of the coefficients, one cannot obtain similar results for the incremental motion of the solid continuum. We shall, therefore, present a method which utilizes the power series expansion of the field quantities.

In the past, the researchers working in this area (Atabek [12], Rachev [14] and Dembay [22]) have dealt with thin walled cylindrical arteries and employed the membrane equations for their analysis. As is well known, however, thin shell theories are valid when the ratio of thickness to mean radius is less than 1/20. But, for most arteries, this ratio varies between $\frac{1}{4} \sim \frac{1}{2}$ which is quite large as compared to allowable limits for membrane theories. This consideration clearly indicates that the membrane or thin shell theories cannot be applied to arteries which appear to be a thick walled cylindrical shell element.

3. Solution to Field Equations

In this section we shall seek a harmonic wave type of solution to the set of differential equations (1) and (12) governing the incremental motions of fluid and solid, respectively. The appropriate form of the field variables should be as

\[
\{p, u, w, \bar{p}, \bar{u}, \bar{w}\} = \{\bar{P}(r), \bar{U}(r), \bar{W}(r), \bar{P}(r), \bar{U}(r), \bar{W}(r)\} \exp \{i\omega t - k_z r\}
\]

where $\omega$ is the angular frequency, $k$ is the wave number and $\bar{P}(r), \ldots, \bar{W}(r)$ are unknown functions standing for the wave amplitudes.

The solutions for $\bar{U}(r), \bar{W}(r), \bar{P}(r)$ satisfying the set of equation (1) may be given by

\[
\begin{align*}
\bar{U}(r) &= k [\bar{A} J_0(kr) + \bar{B} J_0(\omega r)], \\
\bar{W}(r) &= -i[k \bar{A} \bar{J}_0(kr) + \bar{B} \bar{J}_0(\omega r)], \\
\bar{P}(r) &= -i \omega \bar{A} \bar{U}_0(kr)
\end{align*}
\]

where $\bar{A}$ and $\bar{B}$ are two integration constants, $J_0(\omega r)$ and $I_0(\omega r)$ are, respectively, the first kind and modified Bessel functions of order $\omega$.

In order to obtain the solution for the field equations of solid body, we introduce (15) into (12) obtaining the following set of ordinary differential equations

\[
\begin{align*}
\frac{d^2 U}{dx^2} + \frac{U}{1 + x} \frac{dU}{dx} - \frac{\beta_0(x)}{(1 + x)^2} U + \left( \Omega^2 - \xi^2 \Delta(x) \right) U = 0, \\
-ix P + \frac{d^2 W}{dx^2} + \frac{W}{1 + x} \frac{dW}{dx} + \left( \Omega^2 - \xi^2 \Delta(x) \right) W - i\xi \Delta(x) \frac{U}{1 + x} = 0
\end{align*}
\]

In obtaining this set of differential equations we have used the following nondimensionalized quantities

\[
\begin{align*}
\bar{\beta}(r) &= \beta \bar{\beta}(r) \quad (i = 1, 2, \ldots, 8), \\
\bar{U} &= \bar{U} U, \\
\bar{W} &= \bar{W} W, \\
\bar{P} &= \bar{P} P, \\
\bar{\xi} &= \bar{\xi} \Delta
\end{align*}
\]

where $\bar{\xi}$ is the deformed mean radius and $\bar{\xi}$ is the thickness of the arterial wall.
In this case, dropping the exponential factor, the components of stress tensor that we shall need in the boundary conditions take the following form
\[
\sigma_{rr}(x) = \beta P + 2 \left( \alpha_1 \frac{dU}{dx} + \alpha_2 \frac{U}{1 + x} - \zeta \alpha_3 W \right), \quad \sigma_{\theta r}(x) = -P_0 \left( \frac{dW}{dx} - \zeta U \right), \quad \frac{h}{2r} \leq x \leq \frac{h}{2r}.
\]

Hence one can restate the boundary conditions in terms of the new variables as
\[
\left. \begin{array}{l}
\sigma_{rr} \left( -\frac{h}{2r} \right) = \ddot{\sigma}_{rr} \left( \frac{h}{2r} \right), \\
\sigma_{\theta r} \left( -\frac{h}{2r} \right) = \ddot{\sigma}_{\theta r} \left( \frac{h}{2r} \right), \\
\sigma_{rr} \left( \frac{h}{2r} \right) = 0, \\
\sigma_{\theta r} \left( \frac{h}{2r} \right) = 0,
\end{array} \right\}
\]
\[
\left. \begin{array}{l}
\dot{\Omega} U \left( -\frac{h}{2r} \right) = \frac{1}{c_0} \dot{U} \left( \frac{h}{2r} \right), \\
\dot{\Omega} W \left( -\frac{h}{2r} \right) = \frac{1}{c_0} \dot{W} \left( \frac{h}{2r} \right).
\end{array} \right\}
\]

Due to the complex structure of the coefficient functions in (17), it is almost impossible to give a closed form analytical solution to the field equations of the solid continuum. Therefore, here, we shall present a power series approximation to the governing differential equations. To this end, we set
\[
\begin{align*}
\beta_i(x) &= \beta_i^0 + \beta_i^{(1)} x + \beta_i^{(2)} x^2 + \ldots \\
\alpha_i(x) &= \alpha_i^0 + \alpha_i^{(1)} x + \alpha_i^{(2)} x^2 + \ldots \\
P_0 &= P_0^0 + P_0^{(1)} x + P_0^{(2)} x^2 + \ldots \\
U(x) &= U_0 + U_1 x + U_2 x^2 + \ldots \quad W(x) = W_0 + W_1 x + W_2 x^2 + \ldots , \\
P(x) &= P_0 + P_1 x + P_2 x^2 + \ldots
\end{align*}
\]

where the coefficients \( \beta_0^0, \ldots, \alpha_0^0, \ldots, P_0^0 \) are constants to be determined from the series expansion of the corresponding known functions while \( U_0, \ldots, P_4 \) are constants to be determined from the field equations and the boundary conditions.

If (21) is introduced into (17) and the coefficients of the same power of \( x \) are set equal to zero we obtain an infinite set of algebraic equations relating these constants. It is, of course, impossible to get any tractable result by keeping all the terms in the expansion. For the purpose of future analysis we shall, however, retain only the terms up to \( x^4 \). Hence, after this operation equation (17) reduces to
\[
\left. \begin{align*}
P_1 + (\Omega^2 - \beta_2^0 - \xi^2 \beta_2^0) U_0 + \beta_2^0 U_1 + 2 \beta_2^0 U_2 = 0, \\
-\xi \ddot{P}_0 - \xi \beta_0^0 U_0 + \Omega^2 W_0 - \beta_0^0 W_1 + 2 \beta_0^0 W_2 = 0, \\
U_0 + U_1 - \xi \ddot{W}_0 = 0, \\
U_0 + U_1 + 2U_2 - \xi \ddot{W}_1 = 0.
\end{align*} \right\}
\]

To be consistent with (22) the incremental stress components that we shall employ in the boundary conditions must be linear in the variable \( x \). Hence, from (19) we have
\[
\sigma_{rr} = \beta (\sigma_{rr}^0 + x \sigma_{rr}^{(1)}), \quad \sigma_{\theta r} = \beta (\sigma_{\theta r}^0 + x \sigma_{\theta r}^{(1)}),
\]

where
\[
\left. \begin{align*}
\sigma_{rr}^0 &= P_0 + 2(\alpha_1^0 U_0 + \alpha_2^0 U_0 - \xi \alpha_3^0 W_0), \\
\sigma_{\theta r}^0 &= -P_0^0(W_1 - \xi U_0), \\
\sigma_{rr}^{(1)} &= P_1 + 2(\alpha_1^{(1)} U_0 + \alpha_2^{(1)} U_0 + \alpha_3^{(1)} U_0 + 2 \alpha_1^{(1)} U_2 - \xi (\alpha_3^{(1)} W_0 + \alpha_3^{(1)} W_1), \\
\sigma_{\theta r}^{(1)} &= -P_0^0(W_1 - \xi U_0) + P_1^0(W_1 - \xi U_0).
\end{align*} \right\}
\]

These unknown coefficients \( U_i, W_i \) \( (i = 0, 1, 2) \), \( P_j \) \( (j = 0, 1) \), \( A \) and \( B \) are to be determined from the boundary conditions (20). If (23a) is introduced into (14) we get the following set of equations
\[
\left. \begin{align*}
\eta_q \sigma_{rr}^{(0)} - m \sigma_{\theta r}^{(1)} &= \ddot{\sigma}_{rr}, \\
q \sigma_{rr}^{(2)} - m \sigma_{\theta r}^{(1)} &= \ddot{\sigma}_{\theta r}, \\
\sigma_{rr}^{(0)} + \frac{h}{2r} \sigma_{rr}^{(1)} &= 0, \\
\sigma_{\theta r}^{(0)} + \frac{h}{2r} \sigma_{\theta r}^{(1)} &= 0.
\end{align*} \right\}
\]

\[
\left. \begin{align*}
\dot{\Omega} \left( U_0 - \frac{h}{2r} U_1 + \frac{h^2}{4r^2} U_2 \right) - \xi I_1(\xi) A - \xi J_1(\xi) B = 0, \\
2 \left( W_0 - \frac{h}{2r} W_1 + \frac{h^2}{4r^2} W_2 \right) + \xi I_1(\xi) A + \xi J_1(\xi) B = 0
\end{align*} \right\}
\]

where
\[
\left. \begin{align*}
\ddot{\sigma}_{rr} &= \left( \dot{\Omega} + 2 \xi \nu \right) I_0(\xi) - \frac{2 \xi}{1 - \frac{h}{2r}} \nu I_1(\xi) \right] A + 2 \nu \xi \left[ \xi J_0(\xi) - \frac{1}{1 - \frac{h}{2r}} J_1(\xi) \right] B, \\
\ddot{\sigma}_{\theta r} &= -2 \xi \nu I_1(\xi) A - \left[ 2 \xi \nu - \Omega \right] J_1(\xi) B,
\end{align*} \right\}
\]

\[
\left. \begin{align*}
\xi &= \frac{q}{\frac{a}{b}}, \quad A = \frac{abh}{2r^2}, \quad B = \frac{\mu}{\frac{a}{b} \varphi}, \\
\xi &= \frac{q}{\frac{a}{b}}, \quad \zeta = \frac{\varphi}{\frac{a}{b}}, \quad \bar{A} = \frac{c_0 A}{c_0}, \quad \bar{B} = \frac{c_0 B}{c_0}.
\end{align*} \right\}
\]
Introducing (23) into (24) and utilizing (22) and (25) we obtain ten homogeneous algebraic equations relating $U_i$, $W_i$ ($i = 0, 1, 2$), $P_j$ ($j = 0, 1$), $A$ and $B$. In order to have a nonzero solution for these constants, the determinant of the coefficients matrix deduced from these equations must vanish. If this operation is carried out one obtains the dispersion relation as a function of initial deformation and the geometrical features of the artery. The result is so complicated that one cannot infer any tangible result. In what follows we, therefore, shall study the case of thin tubes.

4. Dispersion Relation For Thin Tubes:

Noting the difficulties in evaluating the dispersion equation for a more general case, here we shall present the dispersion relations for thin arteries. Following Rubinow and Keller [13], the thin tube theory can be obtained from (22) — (25) by setting $\left( \frac{h}{r} \right) \rightarrow 0$ provided that $(m)$ remains unchanged. The latter requirement is necessary in order to take into account the inertial effect of the arterial wall. If these restrictions are imposed on the equations (22) to (25) we obtain

$$
\sigma_r^{(0)} = \sigma_r^{(1)} = 0, \quad -2m\sigma_r^{(1)} = \bar{\sigma}_r, \quad -2m\sigma_r^{(2)} = \bar{\sigma}_r.
$$

Hence, the algebraic equations resulting from the boundary conditions take the following form

\[
\begin{align*}
W_0 - \xi U_0 &= 0, \quad P_0 + 2(\gamma_1 U_0 + \xi \gamma_2 W_0) = 0, \\
-2m(\Omega - \xi \gamma_1) U_0 + 2\xi \gamma_1 W_0 + \left[(i \Omega + 2\xi^2) I_0(\xi) - 2\xi I_1(\xi)\right] A + \\
+ 2m \left[J_0(\xi) - J_1(\xi)\right] B &= 0, \\
2i \xi \gamma_1 P_0 + 2\xi \gamma_1 U_0 - 2m \gamma_1 (\Omega^2 - \xi^2 \gamma_4) W_0 - 2m \xi I_0(\xi) A + (\Omega - 2i \xi^2) J_1(\xi) B &= 0, \\
i \Omega U_0 - \xi I_1(\xi) A - \xi J_1(\xi) B &= 0, \quad \Omega W_0 + \xi I_0(\xi) A + \xi J_0(\xi) B &= 0
\end{align*}
\]

where the coefficients $\gamma_i$ ($i = 0, 1, ..., 7$) are defined by

\[
\frac{P_0}{\gamma_0}, \quad \gamma_1 = \alpha_2^{(0)} - \alpha_1^{(0)}, \quad \gamma_2 = \alpha_1^{(0)} - \alpha_0^{(0)}, \quad \gamma_3 = -\beta_0^{(0)} + \beta_1^{(0)} + \beta_2^{(0)} - \beta_3^{(0)}, \\
\gamma_4 = 2\beta_0^{(0)} - \beta_2^{(0)} + \beta_3^{(0)} + 2\alpha_0^{(0)} - \alpha_1^{(0)} - \alpha_2^{(0)} + \alpha_3^{(0)}, \quad \gamma_5 = 2\alpha_0^{(0)} + \alpha_1^{(0)} - \alpha_2^{(0)} - \alpha_3^{(0)}, \\
\gamma_6 = \alpha_0^{(0)} - \alpha_1^{(0)} + \alpha_2^{(0)} - \alpha_3^{(0)}, \quad \gamma_7 = \beta_1^{(0)} + \beta_2^{(0)} + 2\lambda_1^{(0)} - \lambda_2^{(0)} - \lambda_3^{(0)}).
\]

Here in obtaining equations (27) we have utilized equations (22) and eliminated the coefficients $U_i$, $W_i$, $P_i$, $A$ and $B$.

Before studying the dispersion relation for this special case, it might be pertinent to give the explicit expressions of the coefficients appearing in (28). Recalling the definitions of the variable $y$, i.e.,

$$
y^2 = \frac{2r^2}{1 + \frac{r^2}{y^2}} + \frac{R^2}{y^2},
$$

and $r = \frac{r(1 + x)}{x}$, for $(h/2r) \rightarrow 0$, the approximate value of $y$ may be given as follows

$$
y = \frac{y^2}{y - \frac{\lambda}{y}} x + O(x^2); \quad \frac{y^2}{y} \equiv \frac{R}{r}.
$$

Employing this value of $y$ in the expressions of $\alpha_i$, $\beta_i$ and $\gamma_i$ and expanding the result into a power series of $x$ we obtain

\[
\begin{align*}
\gamma_0 &= -\frac{\lambda^2}{y^2} P_0^0, \quad \gamma_1 = \alpha \left(\frac{1}{\lambda} - \frac{y^4}{\lambda^2}\right) F_0 + P_0^0, \quad \gamma_2 = \alpha \left(\frac{y^2}{\lambda^2} - \frac{y^4}{\lambda^4}\right) F_0 - P_0^0, \\
\gamma_3 &= \left[2\alpha \lambda^2 y^2 - \frac{y^4}{\lambda^2} - \frac{1}{y^2}\right] F_0, \\
\gamma_4 &= \left[2\alpha^2 \lambda^2 - \frac{y^4}{\lambda^2} + \alpha^2 \frac{y^4}{\lambda^2}\right] F_0, \quad \gamma_5 = 4P_0^0 + 2\left[\alpha \left(\frac{2}{\lambda} - \frac{y^4}{\lambda^2} - \frac{1}{y^2}\right) + \frac{y^4}{\lambda^2} - \frac{1}{y^2}\right] F_0, \\
\gamma_6 &= \left[2\alpha \frac{1}{\lambda} - \frac{y^4}{\lambda^2} + \frac{y^4}{\lambda^2} - \frac{1}{y^2}\right] F_0 + 2P_0^0, \quad \gamma_7 = 2P_0^0 + \left(\frac{y^2}{\lambda^2} + \frac{y^2}{\lambda^2}\right) F_0.
\end{align*}
\]
Here, in obtaining these expressions we have made use of the following relations

\[
\begin{align*}
\alpha_1^{(0)} &= \alpha \frac{y^4}{\lambda^2} F_0 - F_0^0, \\
\alpha_2^{(0)} &= \alpha \frac{y^2}{\lambda^2} F_0, \\
\alpha_3^{(0)} &= \alpha y^2 F_0, \\
\alpha_4^{(0)} &= \alpha \frac{y^2}{\lambda} F_0 - F_0^0, \\
\beta_0^{(0)} &= \left[ 2\alpha \left( \frac{y^4}{\lambda^2} + \frac{2y^2}{\lambda^2} - 2y^2 \right) + \frac{1}{y^2} \right] F_0 + 2 \left( \alpha_1^{(0)} - \alpha_2^{(0)} \right), \\
\beta_2^{(0)} &= \left[ 2\alpha \left( \frac{y^4}{\lambda^2} + \frac{2y^2}{\lambda^2} - 2y^2 \right) + \frac{1}{y^2} \right] F_0 + 2 \left( \alpha_1^{(0)} - \alpha_2^{(0)} \right), \\
\beta_4^{(0)} &= \lambda^2 F_0, \\
\beta_5^{(0)} &= \frac{y^2}{\lambda^2} F_0, \\
\beta_6^{(0)} &= \frac{1}{y^2} - P_1^0, \\
\beta_7^{(0)} &= \left[ 2\alpha (\lambda^2 - y^2) + \lambda^2 \right] F_0, \\
\beta_8^{(0)} &= 2\alpha \left( \lambda^2 - y^2 \right) F_0 - P_1^0, \\
F_0 &= \exp \left[ \alpha \left( \frac{y^2}{\lambda^2} + \frac{1}{y^2} \lambda^2 - 3 \right) \right], \\
\frac{P_0}{F_0} &= \frac{1}{y^2} \int_y \frac{y}{v} F(v) \, dv - \frac{y^2}{\lambda^2} F_0.
\end{align*}
\]

(31)

In order to have a non-zero solution for the constants appearing in (27), the determinant of the coefficient matrix obtained from these equations must vanish. If this is done we get the following dispersion relation

\[
\begin{align*}
\Omega^2 \left\{ \gamma_0 \left[ m + m \xi^2 (F(\xi) - G(\xi)) \right] + \frac{1}{4} G(\xi) \left[ 1 + m \xi^2 F(\xi) \right] \right\} & - \\
- \Omega^2 \left[ \gamma_0 \left( \gamma_4 + 2 \gamma_2 \right) m \xi^2 + \xi \left[ F(\xi) - G(\xi) \right] \left[ \gamma_0 m \xi^2 (\gamma_4 + 2 \gamma_2 + \gamma_7) - \gamma_0 \gamma_5 m^3 - i \gamma_0 m \nu \Omega - \nu^2 \xi^2 \right] \right] + \\
+ \frac{1}{2} \xi \nu \Omega(\xi) \left[ \gamma_0 m - \gamma_0 \gamma_5 - 2 \gamma_1 \right] m + \frac{1}{2} m F(\xi) \left( \gamma_0 \xi^2 - \gamma_5 - \frac{i \nu}{2} \Omega \left( 4 - F(\xi) \right) \right) \right] + \\
+ \xi \nu (F(\xi) - G(\xi)) \left[ \gamma_0 m^4 (\gamma_4 - 2 \gamma_2) + (\gamma_0 \xi^2 - \gamma_5) (\gamma_4 + 2 \gamma_2) \right] - i \nu \Omega m [\gamma_4 + \gamma_0 \gamma_5 - \gamma_3 + \\
+ 2 \gamma_1 + 2 \gamma_2] = 0
\end{align*}
\]

(32)

where the functions $F(\xi)$ and $G(\xi)$ are defined by

\[
F(\xi) = \frac{2J_1(\xi)}{\xi J_0(\xi)}, \quad G(\xi) = \frac{2J_0(\xi)}{\xi J_0(\xi)}.
\]

(33)

When the initial deformation vanishes, i.e., $\bar{y} = \lambda = 1$, then

\[
F_0 = \gamma_0 = \gamma_2 = 1, \quad P_0^0 = \gamma_4 = 1, \quad P_1^0 = \gamma_3 = 0, \quad \gamma_4 = 2, \quad \gamma_5 = 4, \quad \gamma_6 = -4,
\]

and the dispersion equation reduces to

\[
\begin{align*}
\Omega^2 \left\{ m + m \xi^2 [F(\xi) - G(\xi)] + \frac{1}{4} G(\xi) \left[ 1 + m \xi^2 F(\xi) \right] \right\} & - \\
- \Omega^2 \left[ 4m \xi^2 + \xi \left[ F(\xi) - G(\xi) \right] \left[ 4m^2 (1 + \xi^2) - i \nu \Omega - \nu^2 \xi^2 \right] \right] + \\
+ \frac{1}{2} \xi \nu \Omega(\xi) \left[ \left(-4m + 2m F(\xi) \right) + \frac{i \nu}{2} \Omega \left( 4 - F(\xi) \right) \right] + 12 \xi^4 m^4 [F(\xi) - G(\xi)] = 0
\end{align*}
\]

(34)

If one sets in Rubinow and Keller [13], $E = 3\beta$, $\nu = 0.5$ (Poisson’s ratio) and puts a minus sign in front of the exponential part, their result will exactly be the same with our equation (34). Therefore, one might think of that the dispersion equation (32) is the generalization of Rubinow and Keller [13] which does not take the effect of initial stresses into account.

Before starting the investigation of the more general case it might be pertinent to introduce the nondimensionalized complex phase velocity as

\[
c = \frac{\Omega}{\xi}.
\]

(35)

Decomposing $c$ into real and imaginary parts

\[
c = X + iY
\]

(36)
where, as proposed by Atabek and Lew [8], the wave speed $v$ and the transmission coefficient $\chi$ are defined by

$$v = \frac{X^2 + Y^2}{X}, \quad \chi = \exp\left(-2\pi Y/X\right).$$

(37)

In blood flow problems it is well-known that the Womersley parameter $\omega = (\Omega/v)^{1/2}$ satisfies the condition $\Omega/v \gg \left|\xi^2\right|$ [Küchen [15] and Bauer et al. [23]] and hence we may approximate $\zeta$ by $(-i\Omega/v)^{1/2} (\xi^2 \approx -i\Omega/v)$. Moreover, noting that $|\xi^2|/\Omega \ll 1$ we may neglect the terms in the dispersion relation (32) with factors $v\Omega$ and $v\xi^2$. If this is done the dispersion equation takes the following form

$$c^4(\gamma_0 m + m\xi^2 F(\xi) - G(\zeta)) + \frac{1}{2} G(\zeta) \left(1 + m\xi^2 F(\xi) \right) -$$

$$- c^2(m\gamma_0 \gamma_4 + 2\gamma_2 + (F(\xi) - G(\zeta)) \gamma_0 \gamma_4(\gamma_4 + 2\gamma_2 + \gamma_4 \xi^2 - \gamma_2) +$$

$$+ \frac{1}{2} m G(\zeta) \left[\gamma_0 - \gamma_0 \gamma_3 - 2\gamma_1 + \frac{1}{2} (\gamma_0 \xi^2 - \gamma_3) F(\xi) \right] +$$

$$+ (F(\xi) - G(\zeta)) \gamma_0 m^2 \gamma_4 (\gamma_4 - 2\gamma_1 + \gamma_4 + 2\gamma_2 \gamma_4 \xi^2 - \gamma_2) = 0.$$  

(38)

This equation is quadratic in $c^2$ and it can be solved in terms of the nondimensional frequency and the initial deformations. As this equation reveals, there are two waves propagating in the medium, these waves are dispersive and dissipative, the latter property results from the viscosity of the blood. Before studying this general case by numerical means it might be instructive to investigate some special cases.

(i) **Long wave approximation:** Generally, the wavelength is very large as compared to mean radius of the arteries. Therefore, in this special case $|\xi| \ll 1$ and $F(\xi)$ approaches to unity. Hence, the dispersion equation in this limiting case becomes

$$c^4(\gamma_0 m + \frac{1}{2} G(\zeta)) - c^2(m\gamma_0 \gamma_4 + 2\gamma_2 - \gamma_0 \gamma_0 m^2 (1 - G(\zeta)) + \frac{1}{2} m G(\zeta) \left[\gamma_0 - \gamma_0 \gamma_3 - 2\gamma_1 - \frac{1}{2} \gamma_3 \right]) +$$

$$+ (1 - G(\zeta)) \gamma_0 m^2 \gamma_4 (\gamma_4 - 2\gamma_1 - \gamma_4 \gamma_2 + 2\gamma_2) = 0.$$  

(39)

with

$$\zeta = \frac{\Omega}{(\Omega/v)^{1/2}}.$$  

This quadratic equation still gives two waves which are dispersive and attenuate with distance from the left ventricle.

(ii) **Zero viscosity approximation:** A vanishingly low viscosity limit may be obtained from (38) by setting $\Omega/v \to \infty$, in which case $G(\zeta) \to 0$, and the dispersion relation takes the following form

$$c^4(1 + m\xi^2 F(\xi)) - c^2(\gamma_4 + 2\gamma_2 + m F(\xi) \left[(2\gamma_2 + \gamma_4 + \gamma_4 \xi^2 - \gamma_2) \right) +$$

$$+ m F(\xi) \left[\gamma_0 \gamma_3 - 2\gamma_1 + \gamma_4 + 2\gamma_2 \right] (\gamma_0 \xi^2 - \gamma_3)) = 0.$$  

(40)

The speed of these waves are real but the waves are dispersive. However, in addition to this, if the wavelength is large as compared to mean radius of the artery, from (38) or (40) we have

$$c^4 - c^2(2\gamma_2 + \gamma_4 - m\gamma_3) + m [\gamma_0 (\gamma_3 - 2\gamma_1) - \gamma_0 (\gamma_4 + 2\gamma_2)] = 0.$$  

(41)

The roots of this quadratic equation are given by

$$(c^2)_{1,2} = \frac{1}{2} (2\gamma_2 + \gamma_4 - m\gamma_3) \mp \frac{1}{2} \sqrt{(2\gamma_2 + \gamma_4 - m\gamma_3)^2 - 4m [\gamma_0 (\gamma_3 - 2\gamma_1) - \gamma_0 (\gamma_4 + 2\gamma_2)]}.$$  

(42)

Following Rubenow and Keller [13] and assuming that $m$ is a small parameter, from (42) we obtain,

$$c^2_1 = \gamma_4 + 2\gamma_2 + O(m), \quad c^2_2 = \frac{\gamma_0 (\gamma_3 - 2\gamma_1) - \gamma_0 (\gamma_4 + 2\gamma_2)}{\gamma_4 + 2\gamma_2} m + O(m^2).$$  

(43)

If the initial deformation vanishes then (43) reduces to

$$c^2_1 = 4, \quad c^2_2 = 3m$$  

or, in terms of dimensional quantities

$$v^2_1 = \frac{\beta}{\rho} c^2_1 = \frac{4\beta}{\rho}, \quad v^2_2 = \frac{\beta}{\rho} c^2_2 = \frac{3\beta h}{\rho 2\pi}.$$  

(44)

Recalling that $\beta = E/3$ and $v = 0.5$ (Poisson's ratio) the expressions (44) correspond to Lamb and Young (Moens-Korteweg) modes, respectively.

(iii) **Low frequency limit:** In this special case the Womersley parameter becomes very small and for the small values of the argument $\zeta$, the function $G(z)$ may be approximated by

$$G(z) \approx 1 - \frac{z^2}{8} + \ldots.$$  

(45)
Introducing (45) into (39) and utilizing the definition (37), after a lengthy manipulation, we obtain

\[ v_1 = \left( \frac{\gamma_6(\gamma_6 - 2\gamma_1 - \gamma_6(\gamma_4 + 2\gamma_2))}{4\delta} \right)^{1/2} \frac{m}{v} + \ldots, \]

\[ v_2 = \left( \frac{4m\delta}{1 + 4\gamma_6m} \right)^{1/2} \left\{ 1 + \left( \frac{\Omega}{64} \right)^{1/2} \left[ \frac{4}{1 + 4\gamma_6m} \right] \frac{2[\gamma_2 - \gamma_6(\gamma_2 - 2\gamma_1) - \frac{1}{2} \gamma_6 + 2\gamma_6\gamma_2m]}{\delta} - \frac{\gamma_6[\gamma_6(\gamma_2 - 2\gamma_1) - \gamma_6(\gamma_4 + 2\gamma_2)](1 + 4\gamma_6m)^{1/2}}{\delta^2} \right\} \ldots \]  

(46)

where the parameter \( \delta \) is defined by

\[ \delta = \gamma_6(\gamma_4 + 2\gamma_2) + \frac{1}{2} [\gamma_4 - \gamma_6(\gamma_2 - 2\gamma_1) - \frac{1}{2} \gamma_6]. \]  

Similarly, the transmission coefficients associated with these wave speeds may be given, but for the sake of saving the space we do not give them here. The validity of (46) may also be checked by a numerical evaluation.

(iv) Low viscosity limit: When the viscosity of the blood is low, the Womersley parameter becomes very large (or the inverse of it becomes very small). In this case, for large \( z \) the function \( O(z) \) reads

\[ O(z) = \frac{2}{z} \left( 1 - \frac{1}{2z} + \ldots \right). \]  

(48)

Introducing (48) into (39) and expanding the result into a power series of \( m \) we obtain

\[ v_1 = (\gamma_4 + 2\gamma_2)^{1/2} \left\{ 1 + \frac{1}{2\gamma_6(\gamma_4 + 2\gamma_2)} \left( \frac{v}{2\Omega} \right)^{1/2} \left[ -\gamma_4 + 2\gamma_2 + \gamma_6 - \gamma_6(\gamma_4 - 2\gamma_1) - \frac{\gamma_6(\gamma_2 - 2\gamma_1)}{4(\gamma_4 + 2\gamma_2)} \right] + \ldots \right\}, \]

(49a)

\[ v_2 = \left[ \frac{\gamma_6(\gamma_2 - 2\gamma_1) - \gamma_6(\gamma_4 + 2\gamma_2)}{\gamma_4 + 2\gamma_2} \right]^{1/2} \left\{ 1 - \frac{1}{2\gamma_6(\gamma_4 + 2\gamma_2)} \left( \frac{v}{2\Omega} \right)^{1/2} \times \right\}

\times \left[ 2\gamma_2 + 2\gamma_2 + \gamma_6 - \gamma_6(\gamma_2 - 2\gamma_1) - \frac{\gamma_6(\gamma_2 - 2\gamma_1)}{2(\gamma_4 + 2\gamma_2)} \right] + \ldots \right\}. \]

(49b)

If one sets in \( (49) \left( \frac{v}{2\Omega} \right) \to 0 \), the resulting equations will be the same with that of equation (43).

Now we return to the analysis of the more general equation (39) by numerical means. In one of our previous works (Dembary [18]), employing the experimental results by Simon et al. [34] on canine abdominal artery, whose

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Fig. 1. The variation of primary wave speed with frequency, inner pressure and stretch

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Fig. 2. The variation of secondary wave speed with frequency, inner pressure and stretch.

Fig. 3. The variation of transmission coefficient of primary wave.
Fig. 4. The variation of transmission coefficient of secondary wave

Fig. 5. The variations of phase velocities with pressure and stretch
characteristics are $R_1 = 3.12 \times 10^{-2}$ m, $R_2 = 3.8 \times 10^{-2}$ m, the values of material constants were found to be $\alpha = 1.948$ and $\beta = 9900$ Pa for $\lambda = 1.53$. Utilizing these material and geometrical characteristics in the general dispersion equation the phase velocities and transmission coefficients are evaluated for various inner pressure, stretch ratio and frequency; the results are depicted in Figures 1–6. Figure 1 shows the variation of the phase velocity, of the dominating wave with Womersley parameter and the stretch ratio in the axial direction. It is seen that phase velocity increases with increasing pressure and the stretch ratio (see also Figure 5). The variations of the secondary wave (low phase velocity) with stretch ratio and initial inner pressure are depicted in Figure 2. The wave speed increases with pressure but decreases with axial stretch (see also Figure 5). Furthermore, the speed of the secondary wave becomes insensitive to frequency $\Omega$ after its certain values. The transmission coefficient of the dominating wave decreases with inner pressure while increases with axial stretch; these are illustrated in Figures 3 and 6. Moreover, this coefficient is insensitive to large values of inner pressure. Conversely, the transmission coefficient of the secondary wave increases with pressure while decreasing with stretch ratio (see Figures 4 and 6).

References

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