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NONLINEAR WAVE MODULATION IN MICROPOLAR ELASTIC MEDIA—I. LONGITUDINAL WAVES†

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Abstract—One-dimensional field equations of a homogeneous, isotropic micropolar elastic medium are obtained under the assumption of weak nonlinearity. Nonlinear modulation of longitudinal plane waves propagating in such an infinite medium is investigated. Using the reductive perturbation method, it is shown that the slowly varying complex amplitude of the longitudinal microrotation wave is described by the Nonlinear Schrödinger (NLS) equation. Some special solutions of the NLS equation, namely, nonlinear plane wave, solitary wave and phase jump solutions, are given. The modulational instability of the plane wave solution is also discussed.

1. INTRODUCTION

In the study of the propagation of linear (harmonic) waves in elastic crystals, one observes the coupling of elastic wave modes, through resonance, with a wave mode exhibiting a cutoff. This mode is related to the internal structure of the elastic crystal. This internal structure, or microstructure, may have a different origin, e.g. magnetic (in the case of elastic ferromagnets), electric (in the case of ferroelectric materials) and pure mechanical (in the case of micropolar continua). In the last case, the crystal microstructure is simply characterized by addition of local rigid microrotations. Such a model of continua dates back to the classical works of the Cosserat brothers at the turn of the century [1]. The early theories for this model of continua were referred to as the "constrained" continuum theories. In 1964, Mindlin [2] constructed a microstructure theory of elasticity, Green and Rivlin [3] developed a general multipolar theory, Palmov [4] recapitulated the Cosserat continuum theory, and Eringen and Suhubi [5] and Eringen [6] introduced theories of microelastic solids and fluids. Following these early works, Eringen [7] developed a modern comprehensive formulation for linear micropolar elastic solids. Later, Kafadar and Eringen [8] introduced a fully nonlinear theory of micropolar elastic solids. These theories have been subsequently studied extensively by a large number of researchers. Following Eringen's work, the terminology of micropolar media has been generally adopted. Present-day literature in this area is very extensive (for details, see the comprehensive review article by Eringen and Kafadar [9]).

The theory of micropolar elasticity is concerned with an elastic medium whose constituents, the so-called material points, are allowed to rotate independently without stretch. Hence, the motion of the material points of such a medium will have three additional degrees of freedom associated with local micro-rigid rotations. The fundamental equations of a micropolar elastic medium govern both the usual deformation of the medium (at macroscopic scale) and the evolution of the additional internal degrees of freedom (rotation) involving the microstructural description (at the microscopic scale). Because of the presentation of such an internal structure in the model, the theory of micropolar elasticity is expected to be capable of predicting various phenomena which cannot be predicted by its classical counterpart. For instance, it allows us to observe dispersive wave propagation in the linear approximation [7]. This feature of the model, i.e. its dispersive character, may give rise to various phenomena such as solitary waves if nonlinearity is included. In fact, if either of these effects (i.e. dispersion and nonlinearity) appear alone, there are no solitary wave solutions. When both are present, the competition between the steepening as a result of nonlinearity and the spreading as a result of dispersion favors a travelling wave of constant profile and velocity. Such nonlinear excitations play an

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important role in physics and engineering, and are found in various fields such as nonlinear optics, water waves, plasma physics, etc.

There are already some works on nonlinear wave propagation in micropolar elastic media (for instance, see the works of Dost [10], Maugin and Miled [11] and Erbay and Suhubi [12, 13]). Maugin and Miled [11], by analogy with the domain structure in ferromagnetics and ferroelectrics, were able to introduce the notion of Bloch and Néel walls in the description of micropolar elastic crystals. They showed that, in one dimension, the dynamics of such walls can be represented by solitary waves associated with the microrotation angle and one or two elastic displacement components depending on whether a Bloch or Néel wall is considered. Furthermore, Erbay and Suhubi [12], considering the propagation of small but finite-amplitude weakly nonlinear waves in one dimension, showed that the propagation of long-plane acoustical waves is modelled asymptotically by two coupled Modified Korteweg–de Vries equations. Travelling wave solutions, including solitary wave solutions, of these coupled evolution equations were also studied by the same authors in [13].

In this study we investigate the nonlinear amplitude modulation of plane waves in a micropolar elastic solid, using the reductive perturbation method. Because of the large volume of the study and different characters of the waves considered, the work is presented in two parts. In this part, the basic equations and linear dispersion relations are given, and the modulation of longitudinal plane waves is studied, whereas Part II is devoted to the modulation of transverse waves. In Part I, we first briefly summarize the basic equations of nonlinear micropolar elasticity given by Kafadar and Eringen [8], in Section 2. In the same section, we then derive the one-dimensional nonlinear field equations, using series expansion in both kinematical and constitutive relations. To include small but finite deformations, we keep only quadratic nonlinearities with respect to microrotation and the displacement and microrotation gradients in the resulting equations. Because of the inclusion of nonlinearities in the constitutive equations, the field equations also include some nonlinear material constants along with the linear ones. To see the dispersive character of the medium, we also present the linear dispersion relations of plane waves, which reveal that the longitudinal microrotation and all transverse waves, except the longitudinal displacement wave, are dispersive. In Section 3, the nonlinear field equations are reduced to a system of first-order differential equations and the general procedure of the reductive perturbation method is presented. In Section 4, we show that the nonlinear self-modulation of a longitudinal microrotation wave in micropolar elastic media is governed by the Nonlinear Schrödinger (NLS) equation, resulting from a balance between nonlinearity and dispersion. In the same section, we also give some special solutions of the NLS equation and discuss the modulational instability of the plane wave solution. Concluding comments are given in Section 5.

2. BASIC EQUATIONS AND DISPERSION RELATIONS

A micropolar elastic solid is affected by the local micro-rigid-rotations of the material points. As every particle of a polar medium is assumed to behave like a rigid body, each point of such a medium can translate and independently rotate. Thus, a material point of such a medium has six degrees of freedom, three for translation and three for rotation. The translation is described by the classical motion

$$x_k = x_k(X_K, t) \qquad (k, K = 1, 2, 3) \tag{2.1}$$

where x_k and X_K are the spatial and material Cartesian coordinates of the same material point at times t and $t = 0$, respectively. The rotation is described by a proper orthogonal tensor

$$\chi_{kl} = \chi_{kl}(X_K, t), \qquad \chi_{km}\chi_{lm} = \delta_{kl} \tag{2.2}$$

where δ_{kl} denotes the Kronecker's delta.† Not all components of this rotation tensor are independent by virtue of $(2.2)_2$. Although there are a number of well-known representations of

†Unless otherwise stated, summation convention is valid for repeated indices throughout the text, and Latin indices take the values 1, 2 and 3, and Greek indices the values 2 and 3 only.

orthogonal tensors, for convenience, we use the following representation for this tensor [8]:

$$\chi_{kl} = \exp(\varphi_{kl}), \qquad \varphi_{kl} = -\epsilon_{klm}\varphi_m \tag{2.3}$$

where φ_{kl} and φ_k represent an antisymmetric tensor and the microrotation vector, respectively, and ϵ_{klm} is the alternating tensor. Expansion of $(2.3)_1$ yields the following representation for χ_{kl} in terms of three independent components, φ_m:

$$\chi_{kl} = \cos\theta\delta_{kl} + \frac{1-\cos\theta}{\theta^2}\varphi_k\varphi_l - \frac{\sin\theta}{\theta}\epsilon_{klm}\varphi_m \tag{2.4}$$

where θ is the angle of rotation defined by $\theta = (\varphi_m\varphi_m)^{1/2} \geq 0$.

The material forms of the balance equations with vanishing body forces and body couples are given by (for details, see [8])

Balance of linear momentum:

$$T_{Kk,K} = \rho_0\ddot{x}_k \tag{2.5}$$

Balance of angular momentum:

$$M_{Kk,K} + \epsilon_{klm}x_{l,K}T_{Km} = \rho_0\dot{\sigma}_k \tag{2.6}$$

where a comma denotes partial differentiation with respect to the material coordinates, a dot stands for partial differentiation with respect to time, T_{Kk} and M_{Kk} are the first Piola–Kirchhoff stress and couple stress tensors, respectively, ρ_0 is the density in the reference configuration, $x_{k,K}$ represents the deformation gradient and σ_k is the intrinsic spin vector defined by

$$\sigma_k \equiv j_{kl}\nu_l \tag{2.7}$$

In equation (2.7), j_{kl} denotes the spatial tensor of inertia density and ν_l is the angular velocity vector given by

$$\nu_k = -\frac{1}{2}\epsilon_{klm}\dot{\chi}_{lK}\chi_{mK} \tag{2.8}$$

The use of the representation (2.4) in equations (2.7) and (2.8) with the assumption of spin isotropy (i.e. $j_{kl} = J_0\delta_{kl}$, where J_0 is the inertia density) allows us to express the spin vector σ_k as follows:

$$\sigma_k = J_0\left[\frac{\dot{\theta}}{\theta}\varphi_k + \left(\frac{\varphi_k}{\theta}\right)^{\cdot}\sin\theta + (1-\cos\theta)\epsilon_{klm}\frac{\varphi_l}{\theta}\left(\frac{\varphi_m}{\theta}\right)^{\cdot}\right] \tag{2.9}$$

The constitutive equations for T_{Kk} and M_{Kk} are given as [8]

$$T_{Kk} = \frac{\partial\Sigma}{\partial\mathscr{C}_{KL}}\chi_{kL}, \qquad M_{Kk} = \frac{\partial\Sigma}{\partial\Gamma_{LK}}\chi_{kL} \tag{2.10}$$

where $\Sigma(\mathscr{C}, \Gamma)$ is the strain energy density function, and $\mathscr{C}$ and Γ are the Cosserat deformation and wryness tensors, respectively, defined by

$$\mathscr{C}_{KL} = x_{k,K}\chi_{kL}, \qquad \Gamma_{KL} = \frac{1}{2}\epsilon_{KMN}\chi_{kM,L}\chi_{kN} \tag{2.11}$$

Employing the representation for the microrotation χ given in equation (2.4), the explicit forms of $\mathscr{C}$ and Γ are obtained as

$$\mathscr{C}_{KL} = x_{k,K}\left[\cos\theta\delta_{kL} + (1-\cos\theta)\frac{\varphi_k\varphi_l}{\theta^2}\delta_{lL} - \sin\theta\epsilon_{klm}\frac{\varphi_m}{\theta}\delta_{lL}\right]$$

$$\Gamma_{KL} = \theta_{,L}\frac{\varphi_k}{\theta}\delta_{kK} + \sin\theta\left(\frac{\varphi_k}{\theta}\right)_{,L}\delta_{kK} - (1-\cos\theta)\epsilon_{KMN}\frac{\varphi_k}{\theta}\left(\frac{\varphi_l}{\theta}\right)_{,L}\delta_{kM}\,\delta_{lN} \tag{2.12}$$

For the study of weakly nonlinear waves in the medium, it is appropriate to expand the strain energy function into a power series. We therefore first introduce the strain tensor $\mathscr{E}$ as

$$\mathscr{E}_{KL} = \mathscr{C}_{KL} - \delta_{KL} \tag{2.13}$$

and then expand the resulting strain energy density function, $\Sigma(\mathscr{E}, \Gamma)$, into a power series in terms of the joint invariants of $\mathscr{E}$ and Γ about a natural state. In the expansion we keep only terms up to the third degree in $\mathscr{E}$ and Γ. Further, as we consider isotropic materials in which the full orthogonal group allows reflections (i.e. centrosymmetric), Σ should depend on Γ through the terms including even powers of Γ only. We then obtain the following expansion for Σ:

$$\begin{aligned}\Sigma = {} & \frac{\lambda}{2} I_1^2 + \mu I_2 + (\mu + \kappa) I_4 + \frac{\alpha}{2} I_{10}^2 + \beta I_{11} + \gamma I_{13} + A_1 I_1^3 + A_2 I_1 I_2 + A_3 I_1 I_4 \\ & + \frac{1}{2} A_4 I_1 I_{10}^2 + A_5 I_1 I_{11} + A_6 I_1 I_{13} + A_7 I_3 + A_8 I_5 + A_9 I_7 I_{10} + A_{10} I_8\end{aligned} \tag{2.14}$$

where λ, μ, κ, α, β and γ are linear material constants, $A_i (i = 1, \ldots, 10)$ represent nonlinear material constants of the medium, and $I_1, I_2, \ldots$ are the invariants of $\mathscr{E}$ and Γ and are presented in Appendix A. Similarly, we also expand $\mathscr{E}$ and Γ into power series, and keep only terms up to the second degree in microrotation and the displacement and microrotation gradients. Using these approximate expressions for $\mathscr{E}$ and Γ, given in (A.2), and omitting the terms higher than second degree, the explicit forms of the first Piola–Kirchhoff stress, T_{Kk}, and couple stress, M_{Kk}, tensors for weakly nonlinear micropolar elastic media are calculated and given in Appendix A.

For the study of one-dimensional plane waves, the displacement vector u_k and the microrotation vector φ_k are assumed to be functions of X and t only, i.e.

$$u_k = u_k(X, t), \qquad \varphi_k = \varphi_k(X, t) \tag{2.15}$$

where $X \equiv X_1$ denotes the coordinate along the direction of propagation. It should be noted that here the directions of the material and spatial coordinate axes are chosen to be coincident. In this special case, the stress and couple stress components appearing in the field equations are obtained from equations (A.3) as follows:

$$\begin{aligned}T_{11} = {} & (2\mu + \lambda + \kappa) \frac{\partial u_1}{\partial X} + \frac{1}{2} a_1 \varphi_1^2 + \frac{1}{2} a_2 \varphi_\alpha \varphi_\alpha + a_3 \left(\varphi_3 \frac{\partial u_2}{\partial X} - \varphi_2 \frac{\partial u_3}{\partial X} \right) \\ & + \frac{1}{2} a_4 \left(\frac{\partial u_1}{\partial X} \right)^2 + \frac{1}{2} a_5 \frac{\partial u_\alpha}{\partial X} \frac{\partial u_\alpha}{\partial X} + \frac{1}{2} a_6 \left(\frac{\partial \varphi_1}{\partial X} \right)^2 + \frac{1}{2} a_7 \frac{\partial \varphi_\alpha}{\partial X} \frac{\partial \varphi_\alpha}{\partial X} \\ T_{1\alpha} = {} & (\mu + \kappa) \frac{\partial u_\alpha}{\partial X} - (-1)^\alpha \kappa \varphi_\beta + \frac{1}{2} (a_1 - a_2) \varphi_1 \varphi_\alpha + (-1)^\alpha a_3 \varphi_\beta \frac{\partial u_1}{\partial X} \\ & + a_5 \frac{\partial u_1}{\partial X} \frac{\partial u_\alpha}{\partial X} + a_8 \frac{\partial \varphi_1}{\partial X} \frac{\partial \varphi_\alpha}{\partial X}, \qquad \alpha \neq \beta \\ M_{11} = {} & (\alpha + \beta + \gamma) \frac{\partial \varphi_1}{\partial X} + \frac{1}{2} (2a_{10} + a_8 - a_9) \left(\varphi_2 \frac{\partial \varphi_3}{\partial X} - \varphi_3 \frac{\partial \varphi_2}{\partial X} \right) \\ & + a_6 \frac{\partial u_1}{\partial X} \frac{\partial \varphi_1}{\partial X} + a_8 \frac{\partial u_\alpha}{\partial X} \frac{\partial \varphi_\alpha}{\partial X} \\ M_{1\alpha} = {} & \gamma \frac{\partial \varphi_\alpha}{\partial X} + (-1)^\alpha a_{10} \left(\varphi_\beta \frac{\partial \varphi_1}{\partial X} - \varphi_1 \frac{\partial \varphi_\beta}{\partial X} \right) + (-1)^\alpha a_9 \varphi_\beta \frac{\partial \varphi_1}{\partial X} \\ & + a_7 \frac{\partial u_1}{\partial X} \frac{\partial \varphi_\alpha}{\partial X} + a_8 \frac{\partial \varphi_1}{\partial X} \frac{\partial u_\alpha}{\partial X}, \qquad \alpha \neq \beta\end{aligned} \tag{2.16}$$

where subscripts α and β take the values 2 and 3 only, and the coefficients $a_k (k = 1, \ldots, 10)$,

which are the coefficients of nonlinear terms, are defined by

$$a_1 = 2(A_3 - A_2 - \lambda), \qquad a_2 = 2(A_3 + A_8 - A_2 - A_7 - \lambda - \mu + \kappa/2),$$
$$a_3 = A_2 + A_7 - A_3 - A_8 + \lambda + \mu, \qquad a_4 = 6(A_1 + A_8) + 3(A_2 + A_3) + 2A_7,$$
$$a_5 = A_3 + 2A_8, \qquad a_6 = A_4 + A_5 + A_6 + 2(A_9 + A_{10}), \qquad a_7 = A_6,$$
$$a_8 = A_9 + A_{10}, \qquad a_9 = \alpha + \beta - (A_9 + A_{10}), \qquad a_{10} = \gamma/2 \tag{2.17}$$

Using the above expressions for T_{1k} and M_{1k} [equations (2.16)] and the approximate form of $\dot{\sigma}_k$ [equation (A.4)] in the basic equations [equations (2.5) and (2.6)], we obtain the following one-dimensional nonlinear field equations:

$$(\lambda + 2\mu + \kappa)\frac{\partial^2 u_1}{\partial X^2} + a_1\varphi_1\frac{\partial \varphi_1}{\partial X} + a_2\varphi_\alpha\frac{\partial \varphi_\alpha}{\partial X} + a_3\frac{\partial}{\partial X}\left(\varphi_3\frac{\partial u_2}{\partial X} - \varphi_2\frac{\partial u_3}{\partial X}\right)$$
$$+ a_4\frac{\partial u_1}{\partial X}\frac{\partial^2 u_1}{\partial X^2} + a_5\frac{\partial u_\alpha}{\partial X}\frac{\partial^2 u_\alpha}{\partial X^2} + a_6\frac{\partial \varphi_1}{\partial X}\frac{\partial^2 \varphi_1}{\partial X^2} + a_7\frac{\partial \varphi_\alpha}{\partial X}\frac{\partial^2 \varphi_\alpha}{\partial X^2} - \rho_0\frac{\partial^2 u_1}{\partial t^2} = 0$$

$$(\mu + \kappa)\frac{\partial^2 u_\alpha}{\partial X^2} - (-1)^\alpha\kappa\frac{\partial \varphi_\beta}{\partial X} + \frac{1}{2}(a_1 - a_2)\frac{\partial}{\partial X}(\varphi_1\varphi_\alpha) + (-1)^\alpha a_3\frac{\partial}{\partial X}\left(\varphi_\beta\frac{\partial u_1}{\partial X}\right)$$
$$+ a_5\frac{\partial}{\partial X}\left(\frac{\partial u_1}{\partial X}\frac{\partial u_\alpha}{\partial X}\right) + a_8\frac{\partial}{\partial X}\left(\frac{\partial \varphi_1}{\partial X}\frac{\partial \varphi_\alpha}{\partial X}\right) - \rho_0\frac{\partial^2 u_\alpha}{\partial t^2} = 0, \qquad \alpha \neq \beta$$

$$(\alpha + \beta + \gamma)\frac{\partial^2 \varphi_1}{\partial X^2} + \frac{1}{2}(2a_{10} + a_8 - a_9)\frac{\partial}{\partial X}\left(\varphi_2\frac{\partial \varphi_3}{\partial X} - \varphi_3\frac{\partial \varphi_2}{\partial X}\right)$$
$$+ a_6\frac{\partial}{\partial X}\left(\frac{\partial u_1}{\partial X}\frac{\partial \varphi_1}{\partial X}\right) + a_8\frac{\partial}{\partial X}\left(\frac{\partial \varphi_\alpha}{\partial X}\frac{\partial u_\alpha}{\partial X}\right) - 2\kappa\varphi_1 - a_1\varphi_1\frac{\partial u_1}{\partial X}$$
$$+ \left(a_2 + a_3 - \frac{a_1}{2}\right)\varphi_\alpha\frac{\partial u_\alpha}{\partial X} - \rho_0 J_0\left[\frac{\partial^2 \varphi_1}{\partial t^2} - \frac{1}{2}\left(\varphi_3\frac{\partial^2 \varphi_2}{\partial t^2} - \varphi_2\frac{\partial^2 \varphi_3}{\partial t^2}\right)\right] = 0$$

$$\gamma\frac{\partial^2 \varphi_\alpha}{\partial X^2} + (-1)^\alpha a_{10}\frac{\partial}{\partial X}\left(\varphi_\beta\frac{\partial \varphi_1}{\partial X} - \varphi_1\frac{\partial \varphi_\beta}{\partial X}\right) + (-1)^\alpha a_9\frac{\partial}{\partial X}\left(\varphi_\beta\frac{\partial \varphi_1}{\partial X}\right)$$
$$+ a_7\frac{\partial}{\partial X}\left(\frac{\partial u_1}{\partial X}\frac{\partial \varphi_\alpha}{\partial X}\right) + a_8\frac{\partial}{\partial X}\left(\frac{\partial \varphi_1}{\partial X}\frac{\partial u_\alpha}{\partial X}\right) - 2\kappa\varphi_\alpha - (-1)^\alpha\kappa\frac{\partial u_\beta}{\partial X}$$
$$- \frac{1}{2}(a_1 + 2a_3)\varphi_1\frac{\partial u_\alpha}{\partial X} - a_2\varphi_\alpha\frac{\partial u_1}{\partial X} + (-1)^\alpha a_3\frac{\partial u_1}{\partial X}\frac{\partial u_\beta}{\partial X} - (-1)^\alpha a_8\frac{\partial \varphi_1}{\partial X}\frac{\partial \varphi_\beta}{\partial X}$$
$$- \rho_0 J_0\left[\frac{\partial^2 \varphi_\alpha}{\partial t^2} - \frac{1}{2}(-1)^\alpha\left(\varphi_1\frac{\partial^2 \varphi_\beta}{\partial t^2} - \varphi_\beta\frac{\partial^2 \varphi_1}{\partial t^2}\right)\right] = 0, \qquad \alpha \neq \beta \tag{2.18}$$

which will be used in the next section to study the modulation of waves due to nonlinearity. It must be noted that in the derivation of the above equations we have considered both "geometrical" and "material" nonlinearities. In other words, we have kept nonlinearities with respect to $\partial \mathbf{u}/\partial X$, $\boldsymbol{\varphi}$ and $\partial \boldsymbol{\varphi}/\partial X$ in $\mathscr{E}$ and Γ, and those with respect to $\mathscr{E}$ and Γ in the constitutive equations. One can, for instance, eliminate the material nonlinearities by simply setting $A_i (i = 1, \ldots, 10)$ equal to zero, in which case the coefficients $a_i (i = 1, \ldots, 10)$ will depend on the linear material constants only.

To see the dispersive character of the micropolar elastic medium, we present the linear field

equations corresponding to equations (2.18):

$$c_L^2 \frac{\partial^2 u_1}{\partial X^2} - \frac{\partial^2 u_1}{\partial t^2} = 0$$

$$c_T^2 \frac{\partial^2 u_\alpha}{\partial X^2} - (-1)^\alpha c_0^2 \frac{\partial \varphi_\beta}{\partial X} - \frac{\partial^2 u_\alpha}{\partial t^2} = 0, \qquad \alpha \neq \beta$$

$$b_L^2 \frac{\partial^2 \varphi_1}{\partial X^2} - \omega_0^2 \varphi_1 - \frac{\partial^2 \varphi_1}{\partial t^2} = 0$$

$$b_T^2 \frac{\partial^2 \varphi_\alpha}{\partial X^2} - \omega_0^2 \varphi_\alpha - \frac{1}{2}(-1)^\alpha \omega_0^2 \frac{\partial u_\beta}{\partial X} - \frac{\partial^2 \varphi_\alpha}{\partial t^2} = 0, \qquad \alpha \neq \beta \tag{2.19}$$

where u_k and φ_k represent "small" variations about the natural state and

$$c_L^2 = (\lambda + 2\mu + \kappa)/\rho_0, \qquad c_T^2 = (\mu + \kappa)/\rho_0, \qquad c_0^2 = \kappa/\rho_0$$
$$b_L^2 = (\alpha + \beta + \gamma)/\rho_0 J_0, \qquad b_T^2 = \gamma/\rho_0 J_0, \qquad \omega_0^2 = 2\kappa/\rho_0 J_0 \tag{2.20}$$

Assuming harmonic wave solutions for u_k and φ_k in the form $\exp[i(kX - \omega t)]$, equations (2.19) yield the following dispersion relation:

$$D_1(k, \omega) D_2(k, \omega) [D_3(k, \omega)]^2 = 0 \tag{2.21}$$

where k and ω denote wavenumber and circular frequency, respectively, and D_1, D_2 and D_3 are defined by

$$D_1(k, \omega) = \omega^2 - c_L^2 k^2$$
$$D_2(k, \omega) = \omega^2 - b_L^2 k^2 - \omega_0^2$$
$$D_3(k, \omega) = 2(\omega^2 - c_T^2 k^2)(\omega^2 - b_T^2 k^2 - \omega_0^2) - \omega_0^2 c_0^2 k^2 \tag{2.22}$$

In equation (2.21) D_1 and D_2 represent the dispersion relations corresponding to the longitudinal displacement and longitudinal microrotation modes associated with u_1 and φ_1,

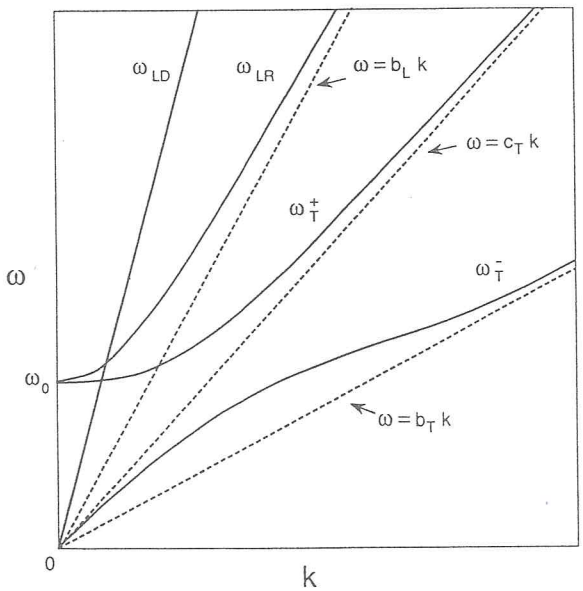

Fig. 1. Dispersion relation in linear micropolar elasticity. ω_{LD}, Longitudinal displacement mode; ω_{LR}, longitudinal microrotation mode; $\omega_T^{\pm}$, coupled transverse displacement and microrotation modes.

respectively. D_3, on the other hand, is the dispersion relation corresponding to the coupled transverse displacement and microrotation modes and satisfied for both the pairs (u_2, φ_3) and (u_3, φ_2). Schematic representation of these dispersion relations are given in Fig. 1. As can be seen from equations (2.22) the longitudinal modes are decoupled from the transverse modes and each other, and the longitudinal microtation mode is strongly dispersive whereas the corresponding displacement mode is nondispersive. The transverse modes corresponding to u_2 and φ_3 (or u_3 and φ_2) are coupled and also dispersive. As those two pairs of transverse waves [i.e. pairs (u_2, φ_3) and (u_3, φ_2)] satisfy the same dispersion relation, one may expect a wave–wave interaction between these transverse modes if nonlinearity is included in the analysis. This will be discussed in detail in the companion paper.

3. NONLINEAR WAVE MODULATION

It is generally difficult to find exact solutions for nonlinear problems. This is particularly valid in the nonlinear dynamic theory of micropolar elasticity because of obvious complexity of the governing equations involved. It is, however, fairly easy to handle nonlinear problems when the nonlinearity is sufficiently weak. In such a case, using the dispersive character of the medium, one can obtain tractable evolution equations arising from a balance between dispersion and nonlinearity. As known from classical linear elasticity, plane harmonic waves in an infinite medium propagate with a constant phase velocity (i.e. the waves are non-dispersive) in contrast to the results of lattice dynamical approaches, which have been verified experimentally. However, the micropolar elasticity theory in the linear approximation gives rise to dispersive branches for harmonic waves in unbounded media. As a consequence of this property, the far-field theory of weakly nonlinear waves, which has been developed considerably in many fields of physics and engineering, can be applied to the micropolar elasticity theory when a balance between nonlinearity and dispersion does hold. By making use of asymptotic methods, under the assumption of weak nonlinearity, the original differential equations for micropolar elastic medium can be reduced to a nonlinear evolution equation or a set of such evolution equations which are similar in form to those previously obtained in water waves, plasma waves, nonlinear optics and other fields, and admit solitary wave solutions. One can therefore expect that solitons, which are not encountered in the classical nonlinear elasticity theory but have received considerable attention in lattice models, may exist in unbounded micropolar elastic solids.

As known, if the amplitude of a wave is small enough, many nonlinear systems admit harmonic wavetrain solutions in which nonlinear terms are small enough to be neglected, and then the amplitude will remain constant in time. If the amplitude of the wave is small but finite, nonlinear terms cannot be neglected and give rise to a variation in the amplitude in both space and time. When the amplitude varies slowly over the period of the oscillation, a stretching transformation allows us to separate the system into a rapidly varying part associated with the oscillation and a slowly varying one such as the amplitude. A formal solution can then be given in the form of an asymptotic expansion, and an equation determining the modulation of the first-order amplitude can be derived. In the next section and the companion paper we investigate respectively how the longitudinal and transverse plane waves in micropolar elastic media are modulated by nonlinear effects. For this purpose, we employ the reductive perturbation method [14].

For convenience we first reduce the nonlinear field equations [equations (2.18)] to a system of first-order differential equations introducing the derivatives of u_k and φ_k as new field variables, i.e.

$$p_k = \frac{\partial u_k}{\partial X}, \qquad v_k = \frac{\partial u_k}{\partial t}, \qquad \psi_k = \frac{\partial \varphi_k}{\partial X}, \qquad w_k = \frac{\partial \varphi_k}{\partial t} \qquad (k = 1, 2, 3) \tag{3.1}$$

Using equations (3.1) in the field equations [equations (2.18)], we obtain the following system

of first-order differential equations:

$$\rho_0\left(c_L^2\frac{\partial p_1}{\partial X}-\frac{\partial v_1}{\partial t}\right)+a_1\varphi_1\psi_1+a_2\varphi_\alpha\psi_\alpha+a_3\left(p_2\psi_3-p_3\psi_2+\varphi_3\frac{\partial p_2}{\partial X}-\varphi_2\frac{\partial p_3}{\partial X}\right)$$
$$+a_4p_1\frac{\partial p_1}{\partial X}+a_5p_\alpha\frac{\partial p_\alpha}{\partial X}+a_6\psi_1\frac{\partial\psi_1}{\partial X}+a_7\psi_\alpha\frac{\partial\psi_\alpha}{\partial X}=0$$

$$\rho_0\left(c_T^2\frac{\partial p_\alpha}{\partial X}-\frac{\partial v_\alpha}{\partial t}\right)-(-1)^\alpha\kappa\psi_\beta+\frac{1}{2}(a_1-a_2)(\psi_1\varphi_\alpha+\varphi_1\psi_\alpha)+(-1)^\alpha a_3\left(p_1\psi_\beta\right.$$
$$\left.+\varphi_\beta\frac{\partial p_1}{\partial X}\right)+a_5\left(p_\alpha\frac{\partial p_1}{\partial X}+p_1\frac{\partial p_\alpha}{\partial X}\right)+a_8\left(\psi_\alpha\frac{\partial\psi_1}{\partial X}+\psi_1\frac{\partial\psi_\alpha}{\partial X}\right)=0,\qquad\alpha\neq\beta$$

$$\rho_0J_0\left(b_L^2\frac{\partial\psi_1}{\partial X}-\frac{\partial w_1}{\partial t}-\omega_0^2\varphi_1\right)-a_1p_1\varphi_1+\left(a_2+a_3-\frac{a_1}{2}\right)p_\alpha\varphi_\alpha$$
$$+\frac{1}{2}(2a_{10}+a_8-a_9)\left(\varphi_2\frac{\partial\psi_3}{\partial X}-\varphi_3\frac{\partial\psi_2}{\partial X}\right)+a_6\left(\psi_1\frac{\partial p_1}{\partial X}+p_1\frac{\partial\psi_1}{\partial X}\right)$$
$$+a_8\left(\psi_\alpha\frac{\partial p_\alpha}{\partial X}+p_\alpha\frac{\partial\psi_\alpha}{\partial X}\right)+\frac{1}{2}\rho_0J_0\left(\varphi_3\frac{\partial w_2}{\partial t}-\varphi_2\frac{\partial w_3}{\partial t}\right)=0$$

$$\rho_0J_0\left(b_T^2\frac{\partial\psi_\alpha}{\partial X}-\frac{\partial w_\alpha}{\partial t}-\omega_0^2\varphi_\alpha\right)-(-1)^\alpha\kappa p_\beta-\frac{1}{2}(2a_3+a_1)\varphi_1p_\alpha-a_2p_1\varphi_\alpha$$
$$+(-1)^\alpha(a_3p_1p_\beta-a_8\psi_1\psi_\beta)+a_7\left(\psi_\alpha\frac{\partial p_1}{\partial X}+p_1\frac{\partial\psi_\alpha}{\partial X}\right)+a_8\left(\psi_1\frac{\partial p_\alpha}{\partial X}+p_\alpha\frac{\partial\psi_1}{\partial X}\right)$$
$$+(-1)^\alpha\left[a_9\left(\psi_1\psi_\beta+\varphi_\beta\frac{\partial\psi_1}{\partial X}\right)+a_{10}\left(\varphi_\beta\frac{\partial\psi_1}{\partial X}-\varphi_1\frac{\partial\psi_\beta}{\partial X}\right)\right.$$
$$\left.+\frac{1}{2}(-1)^\alpha\rho_0J_0\left(\varphi_1\frac{\partial w_\beta}{\partial t}-\varphi_\beta\frac{\partial w_1}{\partial t}\right)\right]=0,\qquad\alpha\neq\beta$$

$$\frac{\partial p_k}{\partial t}-\frac{\partial v_k}{\partial X}=0,\qquad\frac{\partial\psi_k}{\partial t}-\frac{\partial w_k}{\partial X}=0,\qquad\frac{\partial\varphi_k}{\partial t}-w_k=0\qquad(k=1,2,3)\tag{3.2}$$

where p_k, v_k, φ_k, ψ_k and $w_k(k=1,2,3)$ are the unknown field variables.

We now introduce the following scale transformation:

$$\xi=\epsilon(X-\Lambda t),\qquad\tau=\epsilon^2t\tag{3.3}$$

where ξ and τ are slow variables, Λ is a real constant and ϵ is a small parameter measuring the strength of nonlinearity. We then assume that all the field variables, i.e. p_k, v_k, φ_k, ψ_k and w_k $(k=1,2,3)$, have the following series solutions expanded in terms of small parameter ϵ about a natural state:

$$\Theta=\sum_{n=1}^{\infty}\epsilon^n\Theta_n,\qquad\Theta_n=\sum_{l=-\infty}^{+\infty}\Theta_n^l(\xi,\tau)e^{il\theta}\tag{3.4}$$

where Θ was used to represent any of these field varaibles, and $\theta=kX-\omega t$. We have also assumed that the reality conditions $\Theta_n^l=\Theta_n^{-l^*}$ (* denotes complex conjugate) hold. As can be seen from the expansion (3.4), the stretching transformation separates the solution into a rapidly varying part associated with the oscillations and a slowly varying envelope of a carrier wave with these fast oscillations.

As we are dealing with the nonlinear self-modulation of plane waves centered around the frequency ω and the wavenumber k, the coefficients of higher harmonics, except those of first harmonics, are taken as zero for the first-order quantities, i.e. $\Theta_1^l=0$ for $l\neq\pm1$. In this case, the first-order quantities take the following form:

$$\Theta_1=\Theta_1^1(\xi,\tau)e^{i\theta}+\Theta_1^{-1}(\xi,\tau)e^{-i\theta}\tag{3.5}$$

We now want to find out how the slowly varying amplitudes, Θ_1^1, are affected by nonlinearity. Further, to eliminate the self-resonance we assume that the dispersion relation is not satisfied

4. MODULATION OF LONGITUDINAL WAVES

For the modulation of longitudinal waves, we assume that the wavenumber k and the frequency ω satisfy the dispersion relation $D_2(k, \omega) = 0$ and do not satisfy the remaining relations, i.e. $D_1(k, \omega) \neq 0$ and $D_3(k, \omega) \neq 0$. The explicit forms of functions D_1, D_2 and D_3 were given in equations (2.22). In this case, the group velocity is $c_g = b_L^2 k/\omega$. Substituting the slow variables (3.3) and the solution functions (3.4) into the field equations (3.2) and, then equating the coefficients of powers of ϵ to zero we obtain a hierarchy of equations. For $O(\epsilon)$, recalling $\Theta_1^l = 0$ for $l \neq \pm 1$, these equations are

$$
\begin{aligned}
&kc_L^2 p_{11}^1 + \omega v_{11}^1 = 0, \qquad ikc_T^2 p_{\alpha 1}^1 - (-1)^\alpha c_0^2 \psi_{\beta 1}^1 + i\omega v_{\alpha 1}^1 = 0, \qquad \alpha \neq \beta \\
&ikb_L^2 \psi_{11}^1 - \omega_0^2 \varphi_{11}^1 + i\omega w_{11}^1 = 0 \\
&ikb_T^2 \psi_{\alpha 1}^1 - \omega_0^2 \varphi_{\alpha 1}^1 - \frac{(-1)^\alpha}{2} \omega_0^2 p_{\beta 1}^1 + i\omega w_{\alpha 1}^1 = 0, \qquad \alpha \neq \beta \\
&\omega p_{k1}^1 + k v_{k1}^1 = 0, \qquad \omega \psi_{k1}^1 + k w_{k1}^1 = 0, \qquad i\omega \varphi_{k1}^1 + w_{k1}^1 = 0
\end{aligned} \tag{4.1}
$$

Using the conditions $D_1(k, \omega) \neq 0$, $D_2(k, \omega) = 0$ and $D_3(k, \omega) \neq 0$ in equations (4.1), we find

$$
p_{k1}^1 = v_{k1}^1 = 0, \qquad \varphi_{\alpha 1}^1 = \psi_{\alpha 1}^1 = w_{\alpha 1}^1 = 0, \qquad \psi_{11}^1 = ik\varphi_{11}^1, \qquad w_{11}^1 = -i\omega \varphi_{11}^1 \tag{4.2}
$$

It should be noted that all the first-order amplitudes other than those associated with the longitudinal microrotation wave are equal to zero.

Similarly, for $O(\epsilon^2)$, we obtain the following equations:

$$
\begin{aligned}
&\sum_{l=-\infty}^{\infty} \left[il\rho_0 (kc_L^2 p_{12}^l + \omega v_{12}^l) + \sum_{j=-\infty}^{\infty} (a_1 \varphi_{11}^l + a_6 ijk \psi_{11}^l) \psi_{11}^j e^{ij\theta} \right] e^{il\theta} = 0 \\
&\sum_{l=-\infty}^{\infty} [il(kc_T^2 p_{\alpha 2}^l + \omega v_{\alpha 2}^l) - (-1)^\alpha c_0^2 \psi_{\beta 2}^l] e^{il\theta} = 0, \qquad \alpha \neq \beta \\
&\sum_{l=-\infty}^{\infty} \left(b_L^2 \frac{\partial \psi_{11}^l}{\partial \xi} + ilkb_L^2 \psi_{12}^l - \omega_0^2 \varphi_{12}^l + \Lambda \frac{\partial w_{11}^l}{\partial \xi} + il\omega w_{12}^l \right) e^{il\theta} = 0 \\
&\sum_{l=-\infty}^{\infty} \left(ilkb_T^2 \psi_{\alpha 2}^l - \omega_0^2 \varphi_{\alpha 2}^l - \frac{(-1)^\alpha}{2} \omega_0^2 p_{\beta 2}^l + il\omega w_{\alpha 2}^l \right) e^{il\theta} = 0, \qquad \alpha \neq \beta \\
&\sum_{l=-\infty}^{\infty} l(\omega p_{k2}^l + k v_{k2}^l) e^{il\theta} = 0 \\
&\sum_{l=-\infty}^{\infty} \left(\Lambda \frac{\partial \psi_{k1}^l}{\partial \xi} + il\omega \psi_{k2}^l + \frac{\partial w_{k1}^l}{\partial \xi} + ilk w_{k2}^l \right) e^{il\theta} = 0 \\
&\sum_{l=-\infty}^{\infty} \left(\Lambda \frac{\partial \varphi_{k1}^l}{\partial \xi} + il\omega \varphi_{k2}^l + w_{k2}^l \right) e^{il\theta} = 0
\end{aligned} \tag{4.3}
$$

Setting the coefficients of each mode equal to zero, and using the results of $O(\epsilon)$ [equations (4.2)] the solution of equations (4.3) for the first three modes yields

$$
\begin{aligned}
&l = 0: \quad \varphi_{12}^0 = 0, \qquad \psi_{\alpha 2}^0 = 0, \qquad w_{k2}^0 = 0, \qquad p_{\alpha 2}^0 = 2(-1)^\alpha \varphi_{\beta 2}^0, \qquad \alpha \neq \beta \\
&l = 1: \quad p_{k2}^1 = v_{k2}^1 = 0, \qquad \varphi_{\alpha 2}^1 = \psi_{\alpha 2}^1 = w_{\alpha 2}^1 = 0 \\
&\qquad \psi_{12}^1 = ik\varphi_{12}^1 + \frac{\partial \varphi_{11}^1}{\partial \xi}, \qquad w_{12}^1 = -i\omega \varphi_{12}^1 - \Lambda \frac{\partial \varphi_{11}^1}{\partial \xi} \\
&l = 2: \quad \varphi_{k2}^2 = 0, \qquad p_{\alpha 2}^2 = v_{\alpha 2}^2 = \psi_{\alpha 2}^2 = w_{\alpha 2}^2 = 0 \\
&\qquad p_{12}^2 = \frac{(a_6 k^2 - a_1)k^2}{2\rho_0 (c_L^2 k^2 - \omega^2)} (\varphi_{11}^1)^2
\end{aligned} \tag{4.4}
$$

It is not difficult to see that all higher-order terms are equal to zero for $l \geq 3$. The solution of the equations for the first mode ($l = 1$) in equations (4.3) requires that $(\Lambda - c_g)\partial \varphi_{11}^1/\partial \xi = 0$. Obviously, to have a nontrivial solution, the coefficient of $\partial \varphi_{11}^1/\partial \xi$ must vanish. This gives

$\Lambda = c_g$, i.e. the constant Λ is equal to the group velocity of the longitudinal microrotation wave.

We now proceed to the third-order problem, to determine the behaviour of the complex amplitude φ_{11}^1. Here, for $O(\epsilon^3)$, we give only the equations related to the longitudinal displacement and longitudinal microrotation components as the remaining equations are not needed. These equations are

$$\sum_{l=-\infty}^{\infty}\left\{\rho_0\left[c_L^2\left(\frac{\partial p_{12}^l}{\partial\xi}+ilkp_{13}^l\right)+c_g\frac{\partial v_{12}^l}{\partial\xi}+il\omega v_{13}^l\right]+\sum_{j=-\infty}^{\infty}\left[a_1(\psi_{12}^l\varphi_{11}^j\right.\right.$$
$$\left.+\varphi_{12}^l\psi_{11}^j)+a_6\left(\psi_{11}^l\frac{\partial\psi_{11}^j}{\partial\xi}+ik(l+j)\psi_{12}^l\psi_{11}^j\right)\Big]e^{ij\theta}\Big\}e^{il\theta}=0$$
$$\sum_{l=-\infty}^{\infty}\left\{\rho_0J_0\left(b_L^2\frac{\partial\psi_{12}^l}{\partial\xi}+ilkb_L^2\psi_{13}^l-\omega_0^2\varphi_{13}^l+c_g\frac{\partial w_{12}^l}{\partial\xi}+il\omega w_{13}^l-\frac{\partial w_{11}^l}{\partial\tau}\right)\right.$$
$$\left.+\sum_{j=-\infty}^{\infty}p_{12}^l[-a_1\varphi_{11}^j+i(l+j)ka_6\psi_{11}^j]e^{ij\theta}\right\}e^{il\theta}=0$$
$$\sum_{l=-\infty}^{\infty}\left(\frac{\partial\psi_{11}^l}{\partial\tau}-c_g\frac{\partial\psi_{12}^l}{\partial\xi}-il\omega\psi_{13}^l-\frac{\partial w_{12}^l}{\partial\xi}-ilkw_{13}^l\right)e^{il\theta}=0$$
$$\sum_{l=-\infty}^{\infty}\left(\frac{\partial\varphi_{11}^l}{\partial\tau}-c_g\frac{\partial\varphi_{12}^l}{\partial\xi}-il\omega\varphi_{13}^l-w_{13}^l\right)e^{il\theta}=0 \tag{4.5}$$

Solving the first equation in equations (4.5) for $l=0$ we obtain p_{12}^0 as

$$p_{12}^0=\frac{a_1+a_6k^2}{\rho_0(c_g^2-c_L^2)}|\varphi_{11}^1|^2 \tag{4.6}$$

As can easily be seen from the expressions for p_{12}^0 in equation (4.6) and p_{12}^2 in equation (4.4), although there is no longitudinal displacement wave at the first order; the longitudinal microrotation wave produces a longitudinal displacement wave at the second order as a result of nonlinearity. Writing equations (4.5) for $l=1$ and eliminating the higher-order terms ψ_{13}^1 and w_{13}^1 in the resulting expressions, the following NLS equation is obtained for the amplitude of the longitudinal microrotation wave, φ_{11}^1,

$$i\frac{\partial\Phi}{\partial\tau}+\Gamma_L\frac{\partial^2\Phi}{\partial\xi^2}+\Delta|\Phi|^2\Phi=0 \tag{4.7}$$

where $\Phi=\varphi_{11}^1$ and

$$\Gamma_L=\frac{1}{2}\frac{d^2\omega}{dk^2}=\frac{b_L^2\omega_0^2}{2\omega^3},$$
$$\Delta=\frac{1}{2\rho_0^2J_0\omega}\left[\frac{k^2(a_6k^2-a_1)^2}{2(c_L^2k^2-\omega^2)}-\frac{(a_6k^2+a_1)^2}{c_g^2-c_L^2}\right] \tag{4.8}$$

As a consequence of the relations (4.2), this evolution equation is also valid, with slightly different coefficients, for ψ_{11}^1 and w_{11}^1. It must be noted that, in the cases where either $\omega^2=c_L^2k^2$ or $c_g=c_L$, the coefficient Δ becomes infinitely large and then the NLS equation breaks down. Noting that, we have already assumed $\omega^2\neq c_L^2k^2$ [i.e. $D_1(k,\omega)\neq0$] to avoid such a case we further require that the group velocity of the longitudinal microrotation wave be different from the phase velocity c_L of the longitudinal displacement wave, i.e. $c_g\neq c_L$.

4.1 Some special solutions

The NLS equation arises in a wide variety of fields as an equation describing the self-modulation of one-dimensional monochromatic plane waves in dispersive media [15]. The steady-state solutions of the NLS equation, which generally represent wavetrains expressible in terms of the Jacobian elliptic functions, include a bright and a dark envelope soliton, a phase

jump, and a plane wave with a constant amplitude as special cases. The criterion of whether $\Gamma_L\Delta > 0$ or $\Gamma_L\Delta < 0$ is important in determining how given initial data will evolve for long times for the asymptotic wave field governed by the NLS equation. Now let us seek the travelling wave solutions of the NLS equation in the form [16]

$$\Phi(\xi, \tau) = U(\zeta)\exp[i(K\xi - \Omega\tau)], \qquad \zeta = \xi - v_0\tau, \qquad v_0 = \text{const.} \tag{4.9}$$

where $U(\zeta)$ is a real function of ζ. In this case, if $|\Phi|$ approaches a constant U_0 at infinity, the solution is given by a nonlinear plane wave:

$$\Phi(\xi, \tau) = U_0 \exp[i(K\xi - \Omega\tau)] \tag{4.10}$$

where $\Omega = \Gamma_L K^2 - \Delta U_0^2$. In general, the solution can be obtained in terms of Jacobian elliptic functions under the assumption of $v_0 = 2\Gamma_L K$. The explicit functional forms of these solutions will not be given here, except for the following limit cases. Assuming $U \to 0$ and $dU/d\zeta \to 0$ as $|\zeta| \to \infty$, the solution is given for $\Gamma_L\Delta > 0$:

$$U(\zeta) = A \operatorname{sech}\left[\left(\frac{\Delta}{2\Gamma_L}\right)^{1/2} A\zeta\right] \tag{4.11}$$

where $\Omega = \Gamma_L K^2 - \Delta A^2/2$. For $\Gamma_L\Delta < 0$, if $U \to U_0$ and $dU/d\zeta \to 0$ as $\zeta \to \infty$, the solution is

$$U(\zeta) = U_0 \tanh\left[\left(-\frac{\Delta}{2\Gamma_L}\right)^{1/2} U_0\zeta\right] \tag{4.12}$$

where $\Omega = \Gamma_L K^2 - \Delta U_0^2$. These solutions correspond to an envelope solitary wave and a phase jump, respectively. On the other hand, it is well known that the plane wave solution of the NLS equation is modulationally unstable if $\Gamma_L\Delta > 0$ or stable if $\Gamma_L\Delta < 0$.

4.2 Modulational instability

As mentioned in the previous subsection, both the properties of solutions of the NLS equation and the stability of the plane wave solution depend strongly on the sign of $\Gamma_L\Delta$. Therefore it will be interesting to elaborate further on the variation of $\Gamma_L\Delta$ with nonlinear material properties and wavenumber. To this end, let us first write $\Gamma_L\Delta$ in the form

$$\Gamma_L\Delta = h_0^2 \frac{f(q)}{g(q)} \tag{4.13}$$

where

$$h_0^2 = \frac{b_L^6 a_1^2 \omega_0^2}{4\rho_0^2 J_0 (c_L^2 - b_L^2)^3 \omega^4}, \qquad g(q) = 2(q^2 - 1)(q^2 + V_L^2) \tag{4.14}$$

$$f(q) = q^2(q^2 + V_L^2)(nq^2 - V_L^2 + 1)^2 + 2(q^2 - 1)(q^2 + V_L^2 - 1)(nq^2 + V_L^2 - 1)^2$$

In equations (4.14) we have introduced the following dimensionless quantities:

$$q^2 = \frac{c_L^2 - b_L^2}{\omega_0^2} k^2, \qquad V_L^2 = \frac{c_L^2}{b_L^2}, \qquad n = \frac{a_6}{a_1}\frac{\omega_0^2}{b_L^2} \tag{4.15}$$

It should be noted that $q^2 \neq 1$ and $q^2 \neq -V_L^2$ because $D_1(k, \omega) \neq 0$ and $c_g^2 \neq c_L^2$, respectively. Recalling that rotational modes are usually very slow with respect to displacement modes, at least in the long-wavelength limit [11], we will accept $V_L^2 > 1$ (or $c_L^2 > b_L^2$), which implies $h_0^2 > 0$. Further, as we are interested in only real wavenumbers, we will also require that $q^2 \geq 0$. Let us consider the following:

(1) For $q^2 > 1$, in general, it can easily be seen that $g(q) > 0$ and $f(q) > 0$ and consequently $\Gamma_L\Delta > 0$. That is, for $q^2 > 1$, the plane wave is always modulationally unstable and the nonlinear properties of the medium do not affect the stability.

(2) Let us consider the case where $0 \leq q^2 < 1$ for which $g(q) < 0$. When only the geometric nonlinearity is taken into account, $a_6 \equiv 0$ by virtue of equations (2.17) and consequently $n \equiv 0$ in equations (4.14). In this case, it can be shown that $f(q) < 0$ when $0 \leq q^2 < q_c^2$, and $f(q) > 0$

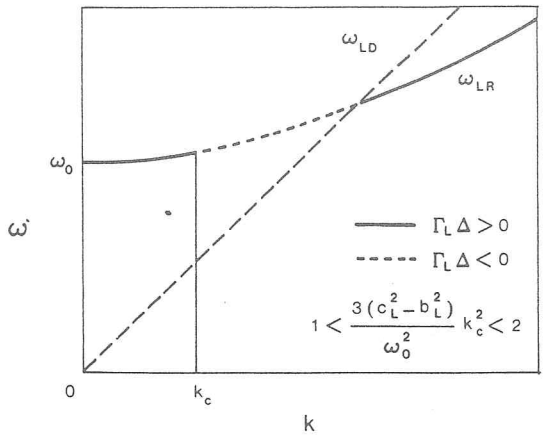

Fig. 2. The regions corresponding to $\Gamma_L\Delta > 0$ and $\Gamma_L\Delta < 0$ for the longitudinal microrotation mode. ω_{LD}; Longitudinal displacement mode; ω_{LR}, longitudinal microrotation mode; k_c, critical wavenumber corresponding to q_c.

when $q_c^2 < q^2 < 1$, where q_c^2 is given by

$$6q_c^2 = -(3V_L^2 - 4) + [(3V_L^2 - 4)^2 + 24(V_L^2 - 1)]^{1/2} \tag{4.16}$$

In other words, when only the geometric nonlinearity is considered, the plane wave is modulationally unstable for $0 \leq q^2 < q_c^2$ and $q^2 > 1$, and stable for $q_c^2 < q^2 < 1$ and for these intervals, the different types of solutions may exist depending on given initial conditions. On the other hand, q_c^2 is an increasing function of V_L^2, which implies that the stability region decreases as V_L^2 increases, and the variation interval of q_c^2 is $1 < 3q_c^2 < 2$. A graphical representation of stable and unstable intervals of the dispersion relation $D_2(k, \omega) = 0$ is given in Fig. 2. For the general case, i.e. $a_6 \neq 0$ (or $n \neq 0$), the variation of $\Gamma_L\Delta$ with q^2 and n has been investigated numerically. The results of this case together with that of the case where $n = 0$ are shown in Fig. 3. It should be noted that the behaviour of $\Gamma_L\Delta$ is strongly affected by the nonlinear mechanical properties of the medium.

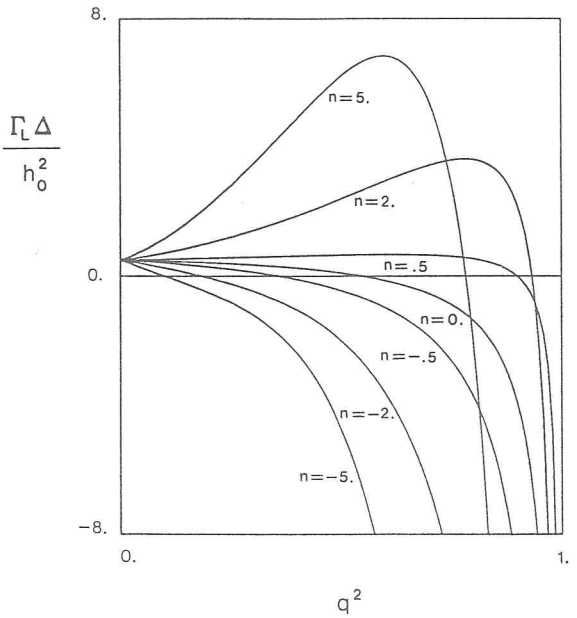

Fig. 3. Variation of $\Gamma_L\Delta$ with q^2 (q is the dimensionless wavenumber) for various values of nonlinear material parameter n when $V_L^2 = 2$.

5. CONCLUSIONS

In this paper we have studied the nonlinear amplitude modulation of longitudinal plane waves in an infinite isotropic micropolar elastic medium. For this purpose, we have derived the one-dimensional field equations of a weakly nonlinear medium so as to include the effects of both material and geometrical nonlinearities. Using the reductive perturbation method, we have shown that the amplitude modulation of a longitudinal microrotation wave is asymptotically governed by the NLS equation. We note that although there is no longitudinal displacement wave at the first-order approximation, nonlinear self-interaction of the longitudinal microrotation wave gives rise to a longitudinal displacement wave at the second order. The analysis shows that the plane wave solution of the NLS equation is modulationally stable within a certain wavenumber interval. In the absence of material nonlinearity, the upper and lower wavenumber values of this interval are obtained analytically. In the general case, i.e. including both material and geometrical nonlinearities, the modulational instability has been examined numerically. The numerical results revealed that the modulational instability is strongly affected by the nonlinear mechanical properties of the medium. Depending upon given initial conditions, it may also be possible to obtain envelope solitary wave solutions outside of this interval.

Nonlinear longitudinal waves propagating in dispersive continuous media have received considerable attention because of their ability to give rise to stable nonlinear structures such as solitary waves. However, in such a medium there also exist transverse waves even for one-dimensional models, and it would be interesting to find out whether stable nonlinear transverse waves are possible. For weakly nonlinear micropolar elastic media, this will be investigated in the second part of this study.

REFERENCES

[1] E. COSSERAT and F. COSSERAT, *Théorie des Corps Deformable.* Hermann (1909).
[2] R. D. MINDLIN, *Archs Rational. Mech. Anal.* **16,** 51 (1964).
[3] A. E. GREEN, and R. S. RIVLIN, *Archs Rational. Mech. Anal.* **17,** 113 (1964).
[4] V. A. PALMOV, *Prikl. Mat. Mekh.* **28,** 401 (1964).
[5] A. C. ERINGEN and E. S. SUHUBI, *Int. J. Engng Sci.* **2,** 189 (1964).
[6] A. C. ERINGEN, *Int. J. Engng Sci.* **2,** 205 (1964).
[7] A. C. ERINGEN, in *Fracture* (Edited by H. Liebowitz), Vol. II. Academic Press, New York (1968).
[8] C. B. KAFADAR and A. C. ERINGEN, *Int. J. Engng Sci.* **9,** 271 (1971).
[9] A. C. ERINGEN and C. B. KAFADAR, in *Continuum Physics* (Edited by A. C. Eringen), Vol. IV. Academic Press, New York (1976).
[10] S. DOST, *Int. J. Engng Sci.* **20,** 769 (1982).
[11] G. A. MAUGIN and A. MILED, *Int. J. Engng Sci.* **24,** 1477 (1986).
[12] S. ERBAY and E. S. SUHUBI, *Int. J. Engng Sci.* **27,** 895 (1989).
[13] S. ERBAY and E. S. SUHUBI, *Int. J. Engng Sci.* **27,** 915 (1989).
[14] A. JEFFREY and T. KAWAHARA, *Asymptotic Methods in Nonlinear Wave Theory.* Pitman, Boston (1982).
[15] A. C. SCOTT, F. Y. F. CHU and D. W. McLAUGHLIN, *Proc. IEEE* **61,** 1443 (1973).
[16] G. L. LAMB, Jr and D. W. McLAUGHLIN, in *Solitons* (Edited by R. K. Bullough and P. J. Caudrey). Springer-Verlag, Berlin (1980).

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APPENDIX A

A.1 The joint invariants of the tensors $\mathscr{E}$ and Γ

$$I_1 = \operatorname{tr}\mathscr{E}, \quad I_2 = \frac{1}{2}\operatorname{tr}\mathscr{E}^2, \quad I_3 = \frac{1}{3}\operatorname{tr}\mathscr{E}^3, \quad I_4 = \frac{1}{2}\operatorname{tr}\mathscr{E}\mathscr{E}^T,$$

$$I_5 = \operatorname{tr}\mathscr{E}^2\mathscr{E}^T, \quad I_6 = \frac{1}{2}\operatorname{tr}\mathscr{E}^2(\mathscr{E}^T)^2, \quad I_7 = \operatorname{tr}\mathscr{E}\Gamma, \quad I_8 = \operatorname{tr}\mathscr{E}\Gamma^2,$$

$$I_9 = \operatorname{tr}\mathscr{E}^2\Gamma, \quad I_{10} = \operatorname{tr}\Gamma, \quad I_{11} = \frac{1}{2}\operatorname{tr}\Gamma^2, \quad I_{12} = \frac{1}{3}\operatorname{tr}\Gamma^3,$$

$$I_{13} = \frac{1}{2}\operatorname{tr}\Gamma\Gamma^T, \quad I_{14} = \operatorname{tr}\Gamma^2\Gamma^T, \quad I_{15} = \frac{1}{2}\operatorname{tr}\Gamma^2(\Gamma^T)^2 \tag{A.1}$$

A.2 Approximate forms of T_{Kk}, M_{Kk} and $\dot{\sigma}_k$

Recalling the expression $x_k = X_K \delta_{kK} + u_k$, where δ_{kK} are the shifters and u_k is the displacement vector, and using equations (2.12) and (2.13), $\mathscr{E}$ and Γ take the following approximate forms:

$$\begin{aligned}\mathscr{E}_{KL} &= u_{k,K}\,\delta_{kL} - \epsilon_{klm}\varphi_m\,\delta_{lL}\,\delta_{kK} - \frac{1}{2}\theta^2\,\delta_{KL} + \frac{1}{2}\varphi_k\varphi_l\,\delta_{kK}\,\delta_{lL} \\ &\quad - \epsilon_{klm}\varphi_m u_{k,K}\,\delta_{lL} + O(.^3) \\ \Gamma_{KL} &= \varphi_{k,L}\,\delta_{kK} - \frac{1}{2}\epsilon_{KMN}\varphi_k\varphi_{l,L}\,\delta_{kM}\,\delta_{lN} + O(.^3)\end{aligned} \tag{A.2}$$

where we have kept the terms only up to second degree in displacement gradients, microrotation and microrotation gradients. Now, first using the approximate strain energy function [equation (2.14)] in the constitutive equations [equations (2.10)] and then using the approximate forms of $\mathscr{E}$ and Γ [equations (A.2)] in the resulting equations, the explicit forms of the first Piola–Kirchhoff stress (T_{Kk}) and couple stress (M_{Kk}) tensors for weakly nonlinear micropolar elastic media are calculated in the form

$$\begin{aligned}T_{Kk} &= \lambda u_{m,M}\,\delta_{mM}\,\delta_{kK} + (\mu+\kappa)u_{k,K} + \mu u_{m,L}\,\delta_{mK}\,\delta_{kL} + \kappa\epsilon_{klm}\varphi_m\,\delta_{lK} \\ &\quad + \left(A_8 - A_7 + A_3 - A_2 + \frac{\kappa}{2} - \lambda - \mu\right)\theta^2\,\delta_{kK} \\ &\quad + \left(A_7 - A_8 + \mu - \frac{1}{2}\kappa\right)\varphi_k\varphi_l\,\delta_{lK} \\ &\quad + (A_3 - A_2 - \lambda)(\epsilon_{klm}\varphi_m u_{n,N}\,\delta_{nN}\,\delta_{lK} + \epsilon_{nlm}\varphi_m u_{n,M}\,\delta_{lM}\,\delta_{kK}) \\ &\quad + (A_8 - A_7 - \mu)(\epsilon_{klm}\varphi_m u_{n,M}\,\delta_{nK}\,\delta_{lM} + \epsilon_{nlm}\varphi_m u_{n,L}\,\delta_{lK}\,\delta_{kL}) \\ &\quad + 3A_1 u_{m,M}u_{n,N}\,\delta_{mM}\,\delta_{nN}\,\delta_{kK} + A_2\left(\frac{1}{2}u_{m,M}u_{n,N}\,\delta_{mN}\,\delta_{nM}\,\delta_{kK}\right. \\ &\quad \left. + u_{m,M}u_{n,L}\,\delta_{mM}\,\delta_{nK}\,\delta_{kL}\right) + A_3\left(\frac{1}{2}u_{m,M}u_{m,M}\,\delta_{kK} + u_{m,M}u_{k,K}\,\delta_{mM}\right) \\ &\quad + \frac{1}{2}\varphi_{m,M}\varphi_{n,N}\,\delta_{kK}(A_4\,\delta_{mM}\,\delta_{nN} + A_5\,\delta_{mN}\,\delta_{nM} + A_6\,\delta_{mn}\,\delta_{MN}) \\ &\quad + A_7 u_{m,M}u_{n,L}\,\delta_{mK}\,\delta_{nM}\,\delta_{kL} + A_8(u_{m,K}u_{k,M}\,\delta_{mM} + u_{n,M}u_{k,M}\,\delta_{nK} \\ &\quad + u_{m,K}u_{m,L}\,\delta_{kL}) + A_9\varphi_{m,M}\varphi_{k,K}\,\delta_{mM} + A_{10}\varphi_{k,M}\varphi_{n,K}\,\delta_{nM} + O(.^3) \\ M_{Kk} &= \alpha\varphi_{m,M}\,\delta_{mM}\,\delta_{kK} + \beta\varphi_{m,L}\,\delta_{mK}\,\delta_{kL} + \gamma\varphi_{k,K} \\ &\quad + \alpha\left(\frac{1}{2}\epsilon_{MLN}\varphi_n\varphi_{l,L}\,\delta_{nM}\,\delta_{lN}\,\delta_{kK} - \epsilon_{kln}\varphi_n\varphi_{m,M}\,\delta_{lK}\,\delta_{mM}\right) \\ &\quad + \beta\left(\frac{1}{2}\epsilon_{KNM}\varphi_m\varphi_{l,L}\,\delta_{mM}\,\delta_{lN}\,\delta_{kL} - \epsilon_{klm}\varphi_m\varphi_{n,L}\,\delta_{lL}\,\delta_{nK}\right) \\ &\quad + \gamma\left(\frac{1}{2}\epsilon_{MLN}\varphi_n\varphi_{l,K}\,\delta_{nM}\,\delta_{lN}\,\delta_{kL} - \epsilon_{klm}\varphi_m\varphi_{l,K}\right) \\ &\quad + A_4 u_{m,M}\varphi_{n,N}\,\delta_{mM}\,\delta_{nN}\,\delta_{kK} + A_5 u_{m,M}\varphi_{n,L}\,\delta_{mM}\,\delta_{nK}\,\delta_{kL} \\ &\quad + A_6 u_{m,M}\varphi_{k,K}\,\delta_{mM} + A_9(u_{k,K}\varphi_{m,M}\,\delta_{mM} + u_{m,M}\varphi_{m,M}\,\delta_{kK} \\ &\quad + \epsilon_{kln}\varphi_n\varphi_{m,M}\,\delta_{mM}\,\delta_{lK} + \epsilon_{lnm}\varphi_m\varphi_{l,M}\,\delta_{nM}\,\delta_{kK}) \\ &\quad + A_{10}(u_{k,M}\varphi_{m,M}\,\delta_{mK} + \epsilon_{klm}\varphi_m\varphi_{n,M}\,\delta_{lM}\,\delta_{nK} \\ &\quad + u_{n,K}\varphi_{n,L}\,\delta_{kL} + \epsilon_{lnm}\varphi_m\varphi_{l,L}\,\delta_{nK}\,\delta_{kL} + O(.^3)\end{aligned} \tag{A.3}$$

where we have neglected the terms higher than second degree. The approximate form of $\dot{\sigma}_k$, on the other hand, is calculated from equation (2.9) as follows:

$$\dot{\sigma}_k = J_0\left(\ddot{\varphi}_k - \frac{1}{2}\epsilon_{klm}\varphi_m\ddot{\varphi}_l\right) + O(.^3) \tag{A.4}$$