AN ASYMPTOTIC THEORY OF THIN HYPERELASTIC PLATES—I. GENERAL THEORY

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Abstract—A large deflection theory of thin hyperelastic plates is developed using an asymptotic analysis of three-dimensional equations of nonlinear elasticity in the reference configuration. The domain occupied by a thin plate is first transformed into a domain of comparable dimensions and then the displacement and stress components are scaled as in the von Kármán theory. The displacement vector and the stress tensor are expanded in terms of powers of an appropriate thickness parameter and a hierarchy of equilibrium equations and boundary conditions are derived following the usual procedure. In a similar fashion the hierarchy of constitutive equations is found for an arbitrary form of strain energy function. The theories corresponding to the lowest two order members in this hierarchy are studied in detail and the zeroth order theory is shown to correspond to the celebrated von Kármán theory. Moreover, it is demonstrated that the effect of physical nonlinearity becomes significant in the first order approximation.

1. INTRODUCTION

There exist several plate theories of various orders in the literature deduced by asymptotic expansions of field variables such as stress tensor and displacement vector in terms of a small thickness parameter of the plate. Although the asymptotic expansion technique is also not free from some ad hoc assumptions, which are notorious characteristics of structures of lower dimensions, it still constitutes a more systematic approach which may naturally yield higher order approximations if one dares to challenge the complexity of the resulting equations.

Works dealing with an asymptotic derivation of the well-known von Kármán equations which accounts for large deflections of thin, linearly elastic plates are first successful attempts towards this direction. Ebiçoğlu and Habip [1] and Habip [2] used simpler forms of the well-known strain-displacement relations as a starting point and applied the method of asymptotic expansion to the governing equations obtained from a variational principle. Assuming the faces of the plate to be free of stress, they have shown that the lowest order approximation corresponds to the von Kármán theory. Ciarlet [3] and Blanchard and Ciarlet [4] provided an excellent and rigorous treatment of the problem by applying asymptotic analysis to the variational formulation of three-dimensional linear elasticity. An approach, which employs directly the method of asymptotic expansion to the nonlinear field equations but leading essentially to same results as given by Ciarlet, is given by Şuhubi [5]. With this approach, it seems that some constraint imposed by the variational formulation can be relaxed and higher order members in the hierarchy become more easily accessible. Later, Şuhubi [6] extended the theory to anisotropic plates having a material symmetry with respect to the middle plane. Following these works Erbay and Şuhubi [7] obtained the nth order field equations of asymptotic approximation for the thin, linearly elastic anisotropic plates and studied the problem of infinite strip under uniform surface load for the lowest two orders. It is necessary to point out that all papers cited consider large displacements and small strains and prefer to work in material coordinates.

In a recent paper, Johnson and Urbanik [8] have used the asymptotic expansion method to develop plate equations representing a generalization of the von Kármán equations for nonlinear anisotropic stress-strain relations. But, in this work, the use of nonsymmetric Piola-Kirchhoff stress tensor increases the number of equations and variables. Moreover, to
avoid an inconsistent theory the nonlinear constitutive equations are approximated by an assumption on the nature of the complementary energy function. At the end, this approxima-
tion is used to characterize the elastic properties of the paper and it is seen that the results of
the theory agree rather well with the experimental data on paper. Later Johnson [9], using a
different scaling from that in the von Kármán theory, investigated the cylindrical deformation
of thin nonlinear elastic plates in plane strain. In this theory, displacement components are
assumed to be of the order of the lateral dimension of the plate and strain components of the
order of the thickness parameter. Finally, Davet [10] attempted to extend the results obtained
by Ciarlet for linear elastic materials to nonlinear ones by employing the same formal
approach. However, since he confined his analysis only to the lowest order terms in the
asymptotic expansion his analysis does not take at all into account the effect of the physical
nonlinearity of the material. Hence his field equations are exactly the same as those
corresponding to a linear elastic material.

The aim of the present work is to develop a generalization of the von Kármán theory for thin
hyperelastic plates by assuming the same type of scaling which produces the classical von
Kármán theory.

In Section 3, the asymptotic expansions of displacement and stress components in terms of
thickness parameter \( \varepsilon \) lead to the hierarchy of governing equations and the theories
corresponding to the first two order members of this hierarchy are studied in detail. The
coefficients appearing in these equations are given for some well-known nonlinear materials
such as Murnaghan and Ko solids and it is shown that the zeroth order theory corresponds to
the well-known von Kármán theory. These equations prove that the zeroth order displacement
field is again a Kirchhoff–Love field. However, the first order displacement field seems to differ
radically from this classical approximation. Moreover, it has been observed in the first order
approximation that the total stress cannot be decomposed into bending and membrane stress
components due to the coupling between bending and stretching. Finally the stress components
in the transverse direction have been determined and the Cauchy stress tensor which represents
the actual state of stress in the deformed plate has also been evaluated.

2. GOVERNING EQUATIONS AND SCALING

We consider a homogeneous and isotropic elastic body which deforms from its natural state
taken as its reference configuration by carrying a particle \( P \) into the spatial position \( p \). The
positions of material and spatial points are denoted by the curvilinear material coordinates \( X^K \)
\((K = 1, 2, 3)\) and the curvilinear spatial coordinates \( x^k \) \((k = 1, 2, 3)\), respectively. The deformation
is given by a set of invertible functions \( x^k = x^k(X^K) \) and the displacement vector by
\( \mathbf{u} = U^K G_K = U_k G_k = u^k g_k = u_k g^k \) where \( G_K \) and \( g_k \) are base vectors along \( X^K \) and \( x^k \)
coordinate curves, respectively. Throughout the analysis the summation convention will be
adopted. For a body occupying a region \( R \) in the reference configuration the equilibrium
equations are in the form (cf. Eringen 11))

\[
T_{.K}^{K} + \rho_0 f^k = 0
\]  
(2.1)

where \( T_{.K}^{K} \) is the first Piola–Kirchhoff stress tensor, \( \rho_0 \) is the initial density, \( f^k \) is the body force
per unit mass and a colon stands for the total covariant differentiation. At the boundary \( \partial R \) of
\( R \), or on a part of it, the given surface tractions \( \tilde{t}^k \) satisfy

\[
T^{K} N_K = \tilde{t}^k
\]  
(2.2)

where \( N_K \) is the unit exterior normal to \( \partial R \). The first and second Piola–Kirchhoff stress tensors,
\( T^{KK} \) and \( T^{KL} \), respectively, are related by

\[
T^{KK} = T^{KL} x^k_{.l}, \quad T^{K} X^L_{.k} = T^{L} X^K_{.k}, \quad T^{KL} = T^{K} X^L_{.k}, \quad T^{KL} = T^{L} K
\]  
(2.3)

where a comma indicates the partial differentiation. On the other hand, the relation between
the deformation and displacement gradients is

\[
x^k_{.K} = (G_{MK} + U_{MK}) g^M_{.k}
\]  
(2.4)
where a semicolon denotes the partial covariant differentiation and \( g^{KK} = G^K \cdot g^K \) is the shifter. If (2.3) and (2.4) are used in (2.1) and (2.2), the equilibrium equations and the boundary conditions take the following forms

\[
\begin{align*}
(T^{KL} + T^{KM} U_{L \beta}^{\beta})_{,K} + \rho_F F^L &= 0, & \text{in } R \\
(T^{KL} + T^{KM} U_{L \beta}^{\beta}) N_K &= \tilde{t}^L, & \text{on } \partial R
\end{align*}
\]  

(2.5)

where \( F^K = f^K g^K \), \( \tilde{t}^K = \tilde{t}^K g^K \). In [8], the asymmetric tensor \( T^{KK} \) and (2.1) and (2.2) are used in the asymptotic analysis. But, in this case, the moments defined on \( T^{KK} \) are not symmetric. However, the number of their independent components is reduced by employing the complicated symmetries in (2.3). This led the authors later to define the moments on \( T^{KL} \) [9].

The constitutive relations of homogeneous isotropic hyperelastic solids are

\[
T^{KL} = \tilde{\varepsilon}_1 G^{KL} + \tilde{\varepsilon}_2 E^{KL} + \tilde{\varepsilon}_3 E^{K}_M E^{ML}
\]  

(2.6)

where \( G^{KL} \) is the metric tensor in the material coordinates, \( E_{KL} \) is the Lagrangian strain tensor and the coefficients \( \tilde{\varepsilon}_k \) \((k = 1, 2, 3)\) are given by

\[
\tilde{\varepsilon}_1 = \frac{\partial \tilde{\Sigma}}{\partial I_E} + I_E \left( \frac{\partial \tilde{\Sigma}}{\partial II_E} + II_E \frac{\partial \tilde{\Sigma}}{\partial III_E} \right), \quad \tilde{\varepsilon}_2 = - \left( \frac{\partial \tilde{\Sigma}}{\partial I_E} + I_E \frac{\partial \tilde{\Sigma}}{\partial III_E} \right), \quad \tilde{\varepsilon}_3 = \frac{\partial \tilde{\Sigma}}{\partial III_E}
\]  

(2.7)

where \( \tilde{\Sigma}(I_E, II_E, III_E) \) is the strain energy function and \( I_E, II_E \) and \( III_E \) are the basic invariants of \( E \) defined by

\[
I_E = \text{tr } E, \quad II_E = \frac{1}{2} \left[ (\text{tr } E)^2 - \text{tr } E^2 \right], \quad III_E = \frac{1}{3} \left[ \text{tr } E^3 - \frac{3}{2} (\text{tr } E)(\text{tr } E^2) + \frac{1}{2} (\text{tr } E)^3 \right]
\]  

(2.8)

It is obvious that if the reference configuration is stress free, the following relation should be satisfied

\[
\frac{\partial \tilde{\Sigma}}{\partial I_E} \bigg|_{I_E=II_E=III_E=0} = 0
\]  

(2.9)

We know that the strain-displacement relations are

\[
2E_{KL} = U_{K;L} + U_{L;K} + U_{M;K} U_{L;M}
\]  

(2.10)

Next let us assume that \( R \) is the domain of a plate which is a cylindrical region of height \( 2h \), i.e. \( R = \Omega \times (-h, h) \) where \( \Omega \), the middle plane of the plate, is an open set with the boundary \( C \) [Fig. 1(a)]. The coordinates are so chosen that \( X^1 \) and \( X^2 \) curves are plane curves parallel to \( \Omega \) and \( X^3 \) is perpendicular to \( \Omega \) which is the plane \( X^3 = 0 \). The upper and lower plane faces and lateral surface of the plate will be denoted by \( \Omega^+ = \Omega \times \{h\} \), \( \Omega^- = \Omega \times \{-h\} \) and \( \Sigma = C \times [-h, h] \), respectively. The position vector \( P \) of an arbitrary point belonging the plate may be written in the form

\[
P = R(X^\gamma) + X^3 A_3
\]

where \( R \) denotes the position vector of an arbitrary point on the middle plane and \( A_3 \), the unit vector normal to the middle plane. Henceforth greek indices will take values 1 and 2 only. Base vectors are now defined by

\[
G_{\sigma}(X^\gamma) = \frac{\partial R}{\partial X^\sigma} = A_{\sigma}(X^\gamma), \quad G_3 = A_3 = A^3
\]

Hence, the components of metric tensor \( G_{KL} \) become

\[
G_{\sigma\beta}(X^\gamma) = A_{\sigma\beta}(X^\gamma), \quad G_{\sigma 3} = 0, \quad G_{33} = 1
\]

where \( A_{\sigma\beta}(X^\gamma) = A_\alpha \cdot A_\beta \). The above form of the metric tensor indicates that the position of the index 3 is immaterial and covariant differentiation with respect to \( X^3 \) coincides with partial differentiation. Furthermore, to simplify some calculations spatial coordinates are so chosen
that \( \mathbf{g}_3 \) and \( \mathbf{G}_3 \) are parallel unit vectors and coordinates \( x^a \) are in a plane which is parallel to \( \Omega \). Therefore we have \( g_{a3} = g_a \cdot G_3 = 0 \) \((a = 1, 2)\), \( g_{3a} = g_3 \cdot G_a = 0 \) and \( g_{33} = g_3 \cdot G_3 = 1 \).

If a characteristic diameter of the region \( \Omega \) is denoted by \( 2L \) then the thickness parameter is defined by \( \varepsilon = h/L \). The thin plate assumption implies that \( \varepsilon \ll 1 \). Now let us consider the following mapping which transforms the domain occupied by a thin plate into a cylindrical domain whose height is of a comparable dimension with the characteristic diameter

\[
X^a = L \xi^a, \quad X^3 = \varepsilon L \xi^3, \quad \xi^3 \in [-1, 1]
\]

Let us assume that \( R(X') = \lambda r(\xi') \). Then the base vectors \( \mathbf{A}_\alpha \) take the form

\[
\mathbf{A}_\alpha(\xi') = \frac{\partial x}{\partial \xi^\alpha}
\]

The upper and lower plane faces and lateral surface of the plate in the dimensionless coordinates are denoted by \( \omega^+ \), \( \omega^- \) and \( \sigma \), respectively, where the middle plane \( \omega \) is an open set in the \( \xi^3 = 0 \) plane with the boundary \( c \) [Fig. 1(b)]. We introduce now nondimensional displacement and stress components in \( \xi \)-coordinates through the relations

\[
U_a(\mathbf{X}) = \varepsilon^2 L u_a(\xi), \quad U^3(\mathbf{X}) = \varepsilon L u^3(\xi) \quad T^{a0}(\mathbf{X}) = T_0 \varepsilon^2 \sigma^{a0}(\xi), \quad T^{33}(\mathbf{X}) = T_0 \varepsilon^4 \sigma^{33}(\xi)
\]

where \( T_0 \) is an appropriately chosen factor of stress dimension. These forms reflect the expected behaviour and ordering of the displacement and stress fields in a thin plate undergoing large transverse deflections as were also assumed in [1]–[7]. The surface tractions \( t_+ \) and \( t_- \) are prescribed on the upper and lower faces, respectively and the stress vector on the lateral surface is denoted by \( \mathbf{f} \). Then the nondimensional surface and body force densities are defined by

\[
t_{a+}(X^\beta) = T_0 \varepsilon^2 \sigma_{a+}^\beta(\xi), \quad t_{a-}(X^\beta) = T_0 \varepsilon^2 \sigma_{a-}^\beta(\xi), \quad X \in \Omega^{+}, \quad \xi \in \omega^+,
\]

\[
\mathbf{r}(X) = T_0 \varepsilon^2 \tau(\xi), \quad \mathbf{r}^3(\mathbf{X}) = T_0 \varepsilon^2 \tau^3(\xi), \quad X \in C, \quad \xi \in c
\]

\[
\rho_0 L F_a(\mathbf{X}) = T_0 \varepsilon^2 \mathbf{f}_a(\xi), \quad \rho_3 L F^3(\mathbf{X}) = T_0 \varepsilon^4 \mathbf{f}^3(\xi)
\]

This scaling reflects the idea that the functions \( g_{a+}, g_{a-}, \tau_a, \tau^3, \mathbf{f}_a, \mathbf{f}^3 \), \( a = 1, 2 \), \( f = 3 \) should be included in the lowest order approximation and external loads tend to zero with the vanishing thickness.

To obtain the equilibrium equations and the boundary conditions in the dimensionless coordinates we must introduce (2.11)–(2.13) into (2.5) to obtain

\[
(\sigma_{a+}^\beta + \varepsilon^2 \sigma_{a+}^\gamma u_{\gamma}^\beta + \varepsilon^2 \sigma_{a+}^{3\beta} u_{3}^\beta)_{,a+} + (\sigma_{3+}^\beta + \varepsilon^2 \sigma_{3+}^\gamma u_{\gamma}^\beta + \varepsilon^2 \sigma_{3+}^{3\beta} u_{3}^\beta)_{,3} + f_{3+} = 0
\]

\[
(\sigma_{a-}^\beta + \varepsilon^2 \sigma_{a-}^\gamma u_{\gamma}^\beta + \varepsilon^2 \sigma_{a-}^{3\beta} u_{3}^\beta)_{,a-} + (\sigma_{3-}^\beta + \varepsilon^2 \sigma_{3-}^\gamma u_{\gamma}^\beta + \varepsilon^2 \sigma_{3-}^{3\beta} u_{3}^\beta)_{,3} + f_{3-} = 0
\]

\[
(\sigma_{3+}^\beta + \sigma_{3+}^{3\beta} u_{3}^\beta)_{,3} + (\sigma_{3-}^\beta + \sigma_{3-}^{3\beta} u_{3}^\beta)_{,3} + f_{3} = 0
\]
and
\[
\begin{align*}
\sigma^{\beta\gamma} + \varepsilon^2 \sigma^{\alpha\gamma} u_{\alpha}^\gamma + \varepsilon^2 \sigma^{\alpha\gamma} u_{\alpha}^\beta = &\ \mp g_{\beta\gamma}, \quad \text{on } \omega^+ \text{ and } \omega^-, \text{ respectively} \\
\sigma^{\alpha\beta} + \sigma^{\alpha\gamma} u_{\gamma}^\beta + \sigma^{\alpha\gamma} u_{\gamma}^\alpha = &\ \mp g_{\alpha\beta},
\end{align*}
\]
where \( \mathbf{n} \) is the exterior unit normal to the lateral surface \( \sigma \) in the transformed domain and derivatives are taken with respect to \( \xi \)-coordinates. Similarly, introducing (2.11)–(2.12) into (2.10) we can express the components of strain tensor, in \( \xi \)-coordinates, as follows:
\[
\begin{align*}
E_{\alpha\beta} &= \frac{1}{2} \varepsilon^2 (u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha} u_{3,\beta}) + \frac{1}{2} \varepsilon^4 u_{\gamma,\alpha} u_{\gamma,\beta} \\
E_{\alpha\gamma} &= \frac{1}{2} \varepsilon (u_{\alpha,\gamma} + u_{\alpha,\gamma} + u_{3,\alpha} u_{3,\gamma}) + \frac{1}{2} \varepsilon^3 u_{\gamma,\alpha} u_{\gamma} \\
E_{33} &= u_{3,3} + \frac{1}{2} u_{3,3} u_{3,3} + \frac{1}{2} \varepsilon^2 u_{3,3} u_{3,3}.
\end{align*}
\]
Equations (2.14)–(2.16) and the constitutive relation (2.6) are the fundamental equations on which the asymptotic analysis will be based.

### 3. ASYMPTOTIC EXPANSION

Inspired by the form of the equations in (2.14), (2.15) and (2.16) we assume that the displacement vector \( \mathbf{u} \) and the stress tensor \( \sigma \) are expandable into asymptotic series in \( \varepsilon \) as follows
\[
\begin{align*}
\mathbf{u}(\xi) &= \sum_{n=0}^{\infty} \varepsilon^{2n} \mathbf{u}(\xi), \\
\sigma(\xi) &= \sum_{n=0}^{\infty} \varepsilon^{2n} \sigma(\xi),
\end{align*}
\]
Substituting (3.1) into the field equations, collecting like powers of \( \varepsilon^2 \) and equating them to zero, successive systems of differential equations can be obtained. Here, to avoid the complexity of the resulting equations, it will be investigated only the lowest two order approximations. Introducing (3.1) into (2.16) and arranging terms in increasing powers of \( \varepsilon \) we readily obtain that
\[
\begin{align*}
(0) E_{\alpha\beta} &= \frac{1}{2} (0) u_{\alpha,\beta} + (0) u_{\beta,\alpha} + (0) u_{3,\alpha} (0) u_{3,\beta}, \\
(1) E_{\alpha\beta} &= \frac{1}{2} (1) u_{\alpha,\beta} + (1) u_{\beta,\alpha} + (0) u_{3,\alpha} (1) u_{3,\beta} + (1) u_{3,\alpha} (0) u_{3,\beta} + (0) u_{\gamma,\alpha} (0) u_{\gamma,\beta}), \\
(0) E_{\alpha\gamma} &= \frac{1}{2} (0) u_{\alpha,\gamma} + (0) u_{\gamma,\alpha} + (0) u_{3,\alpha} (0) u_{3,\gamma}, \\
(1) E_{\alpha\gamma} &= \frac{1}{2} (1) u_{\alpha,\gamma} + (1) u_{\gamma,\alpha} + (0) u_{3,\alpha} (1) u_{3,\gamma} + (1) u_{3,\alpha} (0) u_{3,\gamma} + (0) u_{\gamma,\alpha} (0) u_{\gamma}, \\
(0) E_{33} &= (0) u_{3,3} + \frac{1}{2} (0) u_{3,3} (0) u_{3,3}, \\
(1) E_{33} &= (1) u_{3,3} + (0) u_{3,3} (1) u_{3,3} + \frac{1}{2} (0) u_{3,3} (0) u_{3,3}, \\
(2) E_{33} &= (2) u_{3,3} + (0) u_{3,3} (2) u_{3,3} + \frac{1}{2} (1) u_{3,3} (1) u_{3,3} + (0) u_{3,3} (1) u_{3,3} \ldots
\end{align*}
\]
where we define
\[
\{E_{\alpha\beta}, E_{\alpha\gamma}, E_{33}\} = \sum_{n=0}^{\infty} \varepsilon^{2n} \{E_{\alpha\beta}, E_{\alpha\gamma}, E_{33}\} \varepsilon^{2n}
\]
The explicit forms of the basic invariants in (2.8) are

\[
I_E = E_\alpha^3 + E_3^3
\]
\[
II_E = \frac{1}{2} (E_\alpha^3 E_3^3 - E_3^3 E_\alpha^3) + E_\alpha^3 E_3^3 - E_3^3 E_\alpha^3
\]
\[
III_E = \frac{1}{6} (2E_\alpha^3 E_3^3 E_\alpha^3 + 6E_3^3 E_\alpha^3 E_\alpha^3 + E_\alpha^3 E_3^3 E_\alpha^3 + 3E_\alpha^3 E_3^3 E_\alpha^3)
\]
\[\ldots - 3E_\alpha^3 E_3^3 E_\alpha^3 - 6E_\alpha^3 E_3^3 E_\alpha^3 - 3E_\alpha^3 E_3^3 E_\alpha^3)\]

(3.4)

Thus substituting (3.3) into (3.4) we obtain the expansions

\[
\{I_E, II_E, III_E\} = \sum_{n=0}^{m} (e^{(n)}I_E, e^{(n)}II_E, e^{(n)}III_E) e^{2n}
\]

(3.5)

where

\[
(0)I_E = (0)E_3^3, \quad (1)I_E = (0)E_\alpha^3 + (1)E_3^3, \quad (2)I_E = (1)E_\alpha^3 + (2)E_3^3, \ldots
\]
\[
(0)II_E = (0)E_\alpha^3 (0)E_3^3 - (0)E_\alpha^3 (0)E_3^3,
\]
\[
(1)II_E = \frac{1}{2} [(0)E_\alpha^3 (0)E_3^3 - (0)E_\alpha^3 (0)E_3^3] + (0)E_\alpha^3 (1)E_3^3 + (1)E_\alpha^3 (0)E_3^3 - 2 (0)E_\alpha^3 (1)E_3^3, \ldots
\]
\[
(0)III_E = \frac{1}{2} [(0)E_\alpha^3 (0)E_3^3 - (0)E_\alpha^3 (0)E_3^3] + (0)E_\alpha^3 (0)E_3^3 + (0)E_\alpha^3 (0)E_3^3 + (0)E_\alpha^3 (0)E_3^3, \ldots
\]

(3.6)

In (3.6) only terms required for the first two orders are shown.

Introducing (3.1) into (2.14), the equilibrium equations corresponding to the first two order approximations are obtained in the form

\[
(0)\sigma_\alpha^\beta + (0)\sigma_3^\beta + f^\beta = 0
\]
\[
(0)\sigma_\alpha^3 + (0)\sigma_3^3 + (0)\sigma_\gamma^\alpha (0)u_\gamma^3 + (0)\sigma_\alpha^\alpha (0)u_\alpha^3 + (0)\sigma_3^3 (0)u_3^3,3 + f^3 = 0
\]

(3.7)

and

\[
(1)\sigma_\alpha^\beta + (1)\sigma_3^\beta + (0)\sigma_\gamma^\alpha (0)u_\gamma^3 + (0)\sigma_\alpha^\alpha (0)u_\alpha^3 + (0)\sigma_3^3 (0)u_3^3,3 + (0)\sigma_\gamma^\alpha (0)u_\gamma^3,3 + (0)\sigma_\alpha^\alpha (0)u_\alpha^3,3 + (0)\sigma_3^3 (0)u_3^3,3,3 = 0
\]

(3.8)

respectively. Similarly, using (3.1) in (2.15), the boundary conditions for the zeroth and first order approximations are given by

\[
(0)\gamma^\beta = \mp g_+^\beta,
\]
\[
(0)\sigma_\alpha^\beta (0)u_\alpha^3 + (0)\sigma_3^3 (0)u_3^3,3 + \mp g_+^3 = \mp g_+^3
\]

(3.9)

and

\[
(1)\gamma^\beta + (0)\sigma_\alpha^\alpha (0)u_\alpha^3 + (0)\sigma_3^3 (0)u_3^3 = 0
\]
\[
(1)\sigma_\alpha^\beta (0)u_\alpha^3 + (1)\sigma_\gamma^\gamma (0)u_\gamma^3,3 + (0)\sigma_\alpha^\alpha (0)u_\alpha^3,3 = 0
\]

(3.10)

respectively.

In order to find an asymptotic hierarchy of the constitutive equations we assume that

\[
e_k = \sum_{n=0}^{m} (e_k e^{2n}) (k = 1, 2, 3)
\]

(3.11)
where we defined \( e_k = \frac{\sigma_k}{T_0} \). \((\text{IV})\) \( e_k \) are obtained using (2.7) and the following Taylor’s expansion

\[
| f(E, E, III_E)| = \left( \frac{\partial f}{\partial E} E + \frac{\partial f}{\partial III_E} III_E \right) + \left( \frac{\partial f}{\partial (1)^2} E + \frac{\partial f}{\partial II_E} II_E \right) + \left( \frac{\partial f}{\partial (1)^3} E + \frac{\partial f}{\partial (0)^2} III_E \right) + \left( \frac{\partial f}{\partial (0)^3} III_E \right) + \left( \frac{\partial f}{\partial (0)^4} \right) + \cdots
\]

We shall determine later the explicit forms of \((\text{V})\) \( e_k \) to the required order. For the present let us introduce (2.12), (3.1), (3.3) and (3.11) into (2.6). Since quantities of negative order in asymptotic expansion should vanish, we obtain at once the following relations

\[
\begin{align*}
(-2) \sigma^{33} &= 0 \to (0) e_1 + (0) e_2 + (0) e_3 (0) E_3 (0) E_3^3 = 0 \\
(-1) \sigma^{\alpha \beta} &= 0 \to (0) e_1 = 0 \\
(1) \sigma^{\alpha \beta} &= 0 \to (0) e_2 + (0) e_3 (0) E_3 (0) E_3^2 = 0 \\
(-1) \sigma^{33} &= 0 \to (1) e_1 + (0) e_2 (1) E_3^3 + (1) e_2 (0) E_3^3 + (0) e_3 (2) (0) E_3^3 (1) E_3 \\
&\quad + (0) E_3^3 (0) E_3^2 + (1) e_3 (0) E_3^3 (0) E_3^2 = 0 \\
(0) \sigma^{\alpha \beta} &= (1) e_1 A^{\alpha \beta} + (0) e_2 (0) E_3^{\alpha \beta} + (0) e_3 (0) E_3^{\alpha \beta} \\
(0) \sigma^{\alpha \beta} &= (0) e_2 (1) E_3^3 + (1) e_2 (0) E_3^3 + (0) e_3 (0) E_3^3 + (1) E_3^2 (0) E_3^{\alpha \beta} \\
&\quad + (1) e_3 (0) E_3^3 (0) E_3^{\alpha \beta} \\
(2) \sigma^{33} &= (2) e_1 + (0) e_2 (2) E_3^3 + (1) e_2 (1) E_3^3 + (2) e_2 (0) E_3^3 + (0) e_3 (2) (0) E_3^3 (2) E_3^3 \\
&\quad + (1) E_3^3 (1) E_3^3 + (2) (0) E_3^3 (1) E_3^3 + (0) e_3 (2) (0) E_3^3 (1) E_3^3 \\
&\quad + (1) E_3^3 (0) E_3^3 (0) E_3^{\alpha \beta} \\
(3) \sigma^{33} &= (3) e_1 + (0) e_2 (3) E_3^3 + (1) e_2 (2) E_3^3 + (3) e_2 (1) E_3^3 + (0) e_3 (3) (0) E_3^3 (3) E_3^3 \\
&\quad + (1) E_3^3 (2) E_3^3 + (3) (0) E_3^3 (2) E_3^3 + (0) e_3 (3) (0) E_3^3 (2) E_3^3 \\
&\quad + (1) E_3^3 (1) E_3^3 (0) E_3^{\alpha \beta} \\
&\quad + (1) E_3^3 (0) E_3^3 (0) E_3^{\alpha \beta} \\
&\quad + (1) e_3 (0) E_3^3 (0) E_3^{\alpha \beta} \\
&\quad + (1) E_3^3 (0) E_3^3 (0) E_3^{\alpha \beta}
\end{align*}
\]

In view of (3.11) and (3.12), it follows from (2.7) that the coefficients \((\text{V})\) \( e_k \) are given by

\[
\begin{align*}
(\text{IV}) e_1 &= \frac{\partial \Sigma}{\partial E} e_1 \bigg|_{e=0} + (\text{IV}) I_E \frac{\partial \Sigma}{\partial III_E} e_1 \bigg|_{e=0} \\
(\text{IV}) e_2 &= - \left( \frac{\partial \Sigma}{\partial II_E} e_1 \bigg|_{e=0} + (\text{IV}) I_E \frac{\partial \Sigma}{\partial III_E} e_1 \bigg|_{e=0} \right) \\
(\text{IV}) e_3 &= \frac{\partial \Sigma}{\partial III_E} e_3 \bigg|_{e=0} \\
\end{align*}
\]

where \( \Sigma = \tilde{\Sigma}/T_0 \). If we employ (3.13)\(_2\), (3.14) and \((\text{V}) I_E = (0) E_3^3 \) from (3.6)\(_1\) in (3.13)\(_1\), the following restrictions follow

\[
\begin{align*}
(-2) \sigma^{33} &= 0 \to (0) E_3^3 \frac{\partial \Sigma}{\partial III_E} e_1 \bigg|_{e=0} = 0 \\
(-1) \sigma^{\alpha \beta} &= 0 \to (0) E_3^3 \frac{\partial \Sigma}{\partial III_E} e_1 \bigg|_{e=0} = 0
\end{align*}
\]

Since \( \Sigma \) is an arbitrary function of invariants and the resulting nonlinear plate theory must include the linear theory (3.15) leads to the result

\[
\begin{align*}
(\text{IV}) E_3^3 &= 0, \\
(\text{IV}) E_3^3 &= 0
\end{align*}
\]

Employing (3.16) in (3.6) we see that \((\text{V}) I_E = (0) I_E = (0) III_E = 0 \) and due to (3.5) we have \( I_E = II_E = III_E = 0 \) at \( e = 0 \). Thus the condition (3.13)\(_2\), becomes equivalent to (2.9). We now use (3.16) in the remaining equations (3.13)\(_{4, 5, 6, 7, 8}\) to obtain

\[
\begin{align*}
(-1) \sigma^{33} &= 0 \to (1) e_1 + (0) e_2 (1) E_3^3 = 0 \\
(0) \sigma^{\alpha \beta} &= (1) e_1 A^{\alpha \beta} + (0) e_2 (0) E_3^{\alpha \beta} \\
(0) \sigma^{\alpha \beta} &= (0) e_2 (1) E_3^3 + (1) e_2 (0) E_3^3 + (0) e_3 (2) (0) E_3^3 (1) E_3^3 \\
(0) \sigma^{\alpha \beta} &= (2) e_1 + (0) e_2 (2) E_3^3 + (1) e_2 (1) E_3^3 + (0) e_3 (3) (0) E_3^3 (3) E_3^3 \\
(0) \sigma^{\alpha \beta} &= (3) e_1 + (0) e_2 (3) E_3^3 + (1) e_2 (2) E_3^3 + (0) e_3 (4) (0) E_3^3 (4) E_3^3 \\
(0) \sigma^{\alpha \beta} &= (4) e_1 + (0) e_2 (4) E_3^3 + (1) e_2 (3) E_3^3 + (0) e_3 (5) (0) E_3^3 (5) E_3^3 \\
(0) \sigma^{\alpha \beta} &= (5) e_1 + (0) e_2 (5) E_3^3 + (1) e_2 (4) E_3^3 + (0) e_3 (6) (0) E_3^3 (6) E_3^3 \\
&\quad + (0) e_3 (3) (0) E_3^3 (3) E_3^3
\end{align*}
\]
Hence (3.12) reduces to
\[ f(I_E, II_E, III_E) = f(0, 0, 0) + \left( \frac{\partial f}{\partial I_E} \right)_{I_E = 0} e^2 + \left( \frac{\partial f}{\partial II_E} \right)_{II_E = 0} e^4 + \cdots \] (3.18)

Then, employing the expansion (3.18) in (2.7) we obtain in view of (3.6) and (3.11)
\[ (0)\varepsilon_2 = \Gamma_1, \quad (0)\varepsilon_3 = \Gamma_2, \quad (1)\varepsilon_1 = \Gamma_3 \varepsilon_i^{(0)} E_y^{(0)} + (1) E_3^{(0)} \]
\[ (0)\varepsilon_2 = \Gamma_5 \varepsilon_i^{(0)} E_y^{(0)} E_5 + (0) E_y^{(0)} E_5 + (0) E_y^{(0)} E_5^2 + (1) E_3^{(0)} \]

where the constants \( \Gamma_k \) \( (k = 0, 1, 2, 3, 4) \) are defined by
\[ \Gamma_0 = -\frac{\partial \Sigma}{\partial I_E} \bigg|_{I_E = 0, II_E = 0} \]
\[ \Gamma_1 = -\frac{\partial \Sigma}{\partial II_E} \bigg|_{II_E = 0} \]
\[ \Gamma_2 = -\frac{\partial \Sigma}{\partial III_E} \bigg|_{III_E = 0} \]
\[ \Gamma_3 = -\frac{\partial \Sigma}{\partial I_E} \bigg|_{I_E = 0, II_E = 0} \]
\[ \Gamma_4 = -\frac{\partial \Sigma}{\partial III_E} \bigg|_{III_E = 0} \]

(3.17) and (3.19) clearly show that the stress components \((0)\sigma^{\alpha\beta}\) and \((1)\sigma^{\alpha\beta}\) include the higher order strain components \((1) E_3^{(0)}\) and \((2) E_3^{(0)}\), respectively. Thus, in view of (3.2), they also contain the higher order displacement components \((1) u_{3,3}\) and \((0) u_{3,3}\), respectively. To achieve a consistent asymptotic theory we employ first (3.19) in (3.17) to obtain
\[ (1) E_3^{(0)} = -\frac{\Gamma_0}{\Gamma_0 + \Gamma_1} E_\sigma^{(0)} \] (3.21)

Upon substituting (3.19) into (3.17), expressing \((2) E_{53}\) in \((0) E_{53}\) and utilizing the resulting expression and (3.21) we deduce that \((0)\sigma^{\alpha\beta}\) and \((1)\sigma^{\alpha\beta}\) can be written as
\[ (0)\sigma^{\alpha\beta} = 2\Delta_0 (\Delta_0^{(0)} E_y^{(0)} A^{\alpha\beta} + (0) E^{\sigma^{\alpha\beta}}) \]
\[ (1)\sigma^{\alpha\beta} = 2\Delta_1 (\Delta_0^{(1)} E_y^{(0)} A^{\alpha\beta} + (1) E^{\sigma^{\alpha\beta}}) + \Delta_0^{(0)} \sigma^{\alpha\beta} + \left( \frac{\Delta_3}{2} E_y^{(0)} E_5 + (0) E_y^{(0)} E_5^2 \right) \]
\[ - (0) E_y^{(0)} E_5 + \Delta_4 (0) E_y^{(0)} E_5^2 \]
\[ A^{\alpha\beta} - \Delta_3^2 (0) E_y^{(0)} E^{\sigma^{\alpha\beta}} + \Delta_2^2 (0) E_y^{(0)} E^{\sigma^{\alpha\beta}} \] (3.22)

where \( \Delta_k \) \( (k = 0, 1, 2, 3, 4) \) are the following constants
\[ \Delta_0 = \frac{\Gamma_0}{\Gamma_0 + \Gamma_1}, \quad \Delta_1 = \frac{\Gamma_1}{\Gamma_0 + \Gamma_1}, \quad \Delta_2 = \Gamma_2, \quad \Delta_3 = \frac{\Gamma_3}{\Gamma_0 + \Gamma_1}, \quad \Delta_4 = \left[ (\Gamma_2 - \Gamma_3) - (\Gamma_0 - \Gamma_1) \right] / (\Gamma_0 + \Gamma_1)^3 \] (3.23)

In the above treatment we have not placed any restriction on the form of the strain energy function \( \Sigma \). But, in [8] and [9], to eliminate \( E_{33} \) from the stress–strain relations, they assumed that the non-planar components may be neglected in the complementary energy function (or consequently in the strain energy function).

The values of coefficients \( \Delta_k \) \( (k = 0, 1, 2, 3, 4) \) which are the material constants evaluated as some derivatives of \( \Sigma \) with respect to invariants at the natural state, are given below for some well-known materials. For the Ko [12] solid, choosing the factor \( T_0 = \mu \) where \( \mu \) is a material constant, the strain energy function is defined as
\[ \Sigma = \frac{1}{2} \left[ \frac{H_C}{I_C} + 2\sqrt{H_C} - 5 \right] \] (3.24)
where $I_C$, $II_C$ and $III_C$ are the basic invariants of $C_{KL}$ Green deformation tensor which is defined by $C_{KL} = G_{KL} + 2E_{KL}$ and they are expressible in the form

$$I_C = 3 + 2I_E, \quad II_C = 3 + 4I_E + 4II_E, \quad III_C = 1 + 2I_E + 4II_E + 8III_E$$

(3.25)

In this case, from (3.24), (3.25), (3.20) and (3.23) we find that

$$\Delta_0 = \frac{1}{3}, \quad \Delta_1 = 1, \quad \Delta_2 = -8, \quad \Delta_3 = \frac{4}{3}, \quad \Delta_4 = -\frac{4}{9}$$

(3.26)

The strain energy function for the Murnaghan solid is given by

$$\Sigma = \frac{\lambda + 2\mu}{2\mu} I_E^2 - 2II_E + \frac{l}{\mu} I_E^3 + \frac{m}{\mu} I_EII_E + \frac{n}{\mu} III_E$$

(3.27)

where $\lambda$, $\mu$ and $l$, $m$, $n$ are Lamé and Murnaghan's constants, respectively and $T_0 = \mu$. Similarly we obtain

$$\Delta_0 = \frac{\lambda}{\lambda + 2\mu}, \quad \Delta_1 = 1, \quad \Delta_2 = \frac{n}{\mu}, \quad \Delta_3 = \frac{2(m + n)}{\lambda + 2\mu}, \quad \Delta_4 = [8\mu^2(3l + m) - 4\lambda(\lambda + \mu)(m + n) - \lambda^3 n/\mu]/(\lambda + 2\mu)^3$$

It is obvious from (3.27) that $l = m = n = 0$ or consequently $\Delta_0 = \Delta_2 = \Delta_3 = 0$ for the classical linear stress–strain relations. Moreover, we see that $\Delta_0 = \nu/(1 - \nu)$ where $\nu = \lambda/2(\lambda + \mu)$ is Poisson’s ratio. To reach the results obtained in [7] it suffices merely to substitute $\Delta_2 = \nu/(1 - \nu)$, $\Delta_1 = 1$ and $\Delta_2 = \Delta_3 = \Delta_4 = 0$ into the relevant equations. Obviously the coefficients $\Gamma_2$, $\Gamma_3$ and $\Gamma_4$ (or $\Delta_2$, $\Delta_3$ and $\Delta_4$) represent the effect of the physical nonlinearity [cf. (3.20)]. We elicit from (3.22) that the physical nonlinearity becomes effective only at the first order approximation.

Next, we consider the restrictions placed by (3.16) on the form of the displacement field. From $(0)^E = 0$ and (3.2) we see that either $(0)^u^3 = 0$ or $(0)^u_3 = 0$ should be satisfied. Considering the fact that the linear theory should be included in a nonlinear plate theory we can discard the latter choice. Thus, we obtain that

$$(0)^u^3 = (0)^W(\xi^1, \xi^2)$$

(3.28)

which implies that zeroth order deflection $(0)^u^3$ is independent of $\xi$ (henceforth we use $\xi$ in place of $L^3$ for convenience). Similarly, if we take into account (3.2) and (3.28), $(0)^E = 0$ requires that $(0)^u_{\alpha,3} = -(0)^W_{,\alpha}$ which yields in-plane components of the zeroth order displacement vector as

$$(0)^u_{\alpha} = (0)^u_{\alpha} - \xi (0)^W_{,\alpha}$$

(3.29)

where $(0)^u_{\alpha} = (0)^u_{\alpha}(\xi^1, \xi^2)$. This proves that the lowest order displacement field is a Kirchhoff–Love field. Now let us write the condition (3.21) explicitly by using (3.2) and the above results. Then we see that $(1)^u_{3,3}$ is expressible as

$$(1)^u_{3,3} = -\Delta_0 (0)^u_{,\alpha} - \xi (0)^W_{,\alpha} - \frac{1}{2} (1 + \Delta_0) (0)^W_{,\alpha} (0)^W_{,\alpha}$$

(3.30)

which integrates to

$$U_3 = U_3(\xi^1, \xi^2) + \int U_3(\xi^1, \xi^2, \xi)$$

(3.31)

where $(1)^U_3$ is the integral of the right hand side of (3.30):

$$U_3 = -\Delta_0 \xi (0)^u_{,\alpha} - \frac{1}{2} (0)^W_{,\alpha} - \frac{1}{2} (1 + \Delta_0) \xi (0)^W_{,\alpha}$$

(3.32)
Similarly, utilizing (3.17), (3.19), (3.2) and the above results, we deduce \((1)u_{a,3}\) from \((0)\sigma_{a3}\) as follows
\[
(1)u_{a,3} = -(1)W_{i,a} - (1)U_{3,a} - (0)W_{i,a} (1)U_{3,3} + (0)W_{i,a} (0)U_{3,a} - \zeta (0)W_{i,a} + \frac{1}{\Delta_1} (0)\sigma_{a3}
\]  
(3.33)

The integration of this equation with respect to \(\zeta\) gives
\[
(1)u_{a} = (1)u_{a} - \zeta (1)W_{i,a} + (1)U_{a}
\]  
(3.34)

where \((1)u_{a} = (1)u_{a} (\xi_1, \xi_2, \xi_3)\), \((1)U_{a} = (1)U_{a} (\xi_1, \xi_2, \xi_3)\) and the form of \((1)U_{a}\) will be given later after having determined \((0)\sigma^{a\beta}\). Nevertheless we consider that \((1)U_{i}\) and \((1)U_{a}\) are known functions of the zeroth order terms for the first order equations. On the other hand, the functions \((1)u_{a}\) and \((1)w_{i}\) will be determined as the solution of the first order field equations. Moreover, from (3.31) and (3.34), it is obvious that the first order displacement field is not a Kirchhoff–Love field.

For the sake of simplicity we now define
\[
(k)\theta_{a\beta}(\xi_1, \xi_2) = (k)u_{a;\beta} + (k)u_{\beta,a} + \sum_{i=1}^{k} (k)w_{i,a} (k-\iota)w_{\beta} \quad k = 0, 1
\]  
(3.35)

Thus, employing (3.28)–(3.35) in (3.2) we can express \((k)E_{a\beta}\) \((k = 0, 1)\) as
\[
(k)E_{a\beta} = \frac{1}{2} (k)\theta_{a\beta} - \zeta (k)w_{a\beta} + (k)\tilde{E}_{a\beta}
\]  
(3.36)

where \((0)\tilde{E}_{a\beta} = 0\) and \((1)\tilde{E}_{a\beta}\) which contains only the zeroth order terms is given by
\[
(1)\tilde{E}_{a\beta} = \frac{1}{2} [(1)U_{a\beta} + (1)U_{\beta,a} + (0)w_{a\beta} (1)U_{3,\beta} + (0)w_{\beta,a} (1)U_{3,a} + (0)\gamma_{\alpha\beta} (0)\gamma_{\alpha\beta}
\]
\[= \zeta (0)w_{a\beta} (0)\gamma_{\alpha\beta} + (0)\gamma_{\alpha\beta} (0)w_{a\beta} + \xi^2 (0)w_{a\beta} (0)\gamma_{\alpha\beta} + (0)\gamma_{\alpha\beta} (0)w_{a\beta}]
\]  
(3.37)

Now, using (3.22) we can express the stress components \((0)\sigma^{a\beta}\) and \((1)\sigma^{a\beta}\) as follows
\[
(k)\sigma^{a\beta} = \Delta_1 (k)\theta^{a\beta} + \Delta_0 (k)\gamma^{a\beta}A^{a\beta} - 2\zeta (k)\gamma^{a\beta} + \Delta_0 (k)\gamma^{a\beta}A^{a\beta}) + (k)\tilde{\sigma}^{a\beta}, \quad k = 0, 1
\]  
(3.38)

where \((0)\tilde{\sigma}^{a\beta} = 0\) and \((1)\tilde{\sigma}^{a\beta}\) which is, in principle, a known function of the zeroth order terms is obtained by substituting \((1)E_{a\beta}\) in lieu of \((1)E_{a\beta}\) in (3.22). Obviously the in-plane Piola–Kirchhoff stresses vary linearly along the thickness of the plate for the zeroth order approximation. Let us define the following zeroth and first order moments of the stress field related to the first two order approximations
\[
\{(k)n_{a\beta}, (k)\tilde{n}_{a\beta}\} = \int_{-1}^{1} (k)\sigma^{a\beta} (k)\tilde{\sigma}^{a\beta} d\xi, \quad \{(k)m_{a\beta}, (k)\tilde{m}_{a\beta}\} = \int_{-1}^{1} (k)\sigma^{a\beta} (k)\tilde{\sigma}^{a\beta} \zeta d\xi, \quad k = 0, 1
\]
\[
\{(k)q_{a}, (0)\tilde{q}_{a}\} = \int_{-1}^{1} (k)\tilde{\sigma}^{a\beta}, \quad \{(0)n_{33}, (0)m_{33}\} = \int_{-1}^{1} (0)\sigma^{33}, \quad \zeta (0)\sigma^{33} d\xi
\]  
(3.39)

where \((0)\tilde{n}_{a\beta} = (0)\tilde{m}_{a\beta} = 0\). Utilizing (3.38) in these definitions we find that, for \(k = 0, 1\),
\[
(k)n_{a\beta} = 2\Delta_1 (k)\theta^{a\beta} + \Delta_0 (k)\gamma^{a\beta}A^{a\beta} + (k)\tilde{n}_{a\beta}, \quad (k)m_{a\beta} = \frac{4}{3} \Delta_1 (k)\gamma^{a\beta} + \Delta_0 (k)\gamma^{a\beta}A^{a\beta} + (k)\tilde{m}_{a\beta}
\]  
(3.40)

Thus the stress components \((k)\sigma^{a\beta}\) can be expressed as
\[
(k)\sigma^{a\beta} = \frac{1}{2} (k)n_{a\beta} - (k)\tilde{n}_{a\beta} + \frac{3}{2} \zeta (k)m_{a\beta} - (k)\tilde{m}_{a\beta} + (k)\tilde{\sigma}^{a\beta}, \quad k = 0, 1
\]  
(3.41)

This relation shows that, for the zeroth order approximation, the total stress can be decomposed into bending and membrane stress components.
Let us integrate (3.7)–(3.8) and \( \zeta \) times (3.7)_1 and (3.8), with respect to \( \zeta \) on the interval \([-1, 1]\). Using the definitions (3.39) and the boundary conditions (3.9)_1,2 and (3.10)_1,2 we find the following moment equations which are the governing equations for two dimensional fields, for \( k = 0, 1 \),

\[
(k)_H^{\alpha \beta} + (k)_T^\beta = 0
\]

\[
(k)_q^{\alpha} + \sum_{i=0}^{k} \left[ (0)_m^{\alpha \gamma} (k-\gamma)_W^{\gamma \alpha} - (0)_T^\gamma (k-\gamma)_W^{\gamma \alpha} \right] + (k)p = 0
\]

\[
(k)_m^{\alpha \beta} = (k)_q^{\beta} + (k)_g^{\beta} = 0
\]

(3.42)

where \((0)_T^\beta\), \((0)_g^\beta\) and \((0)_p\) are defined as

\[
(0)_T^\beta = g_\alpha^\beta + g_\beta^\alpha + \int_{-1}^{1} f^\beta d\xi, \quad (0)_g^\beta = g_\alpha^\beta - g_\beta^\alpha + \int_{-1}^{1} \xi f^\beta d\xi,
\]

\[
(0)_p = g_\alpha^\alpha + g_\beta^\beta + \int_{-1}^{1} f^3 d\xi
\]

(3.43)

and \((1)_T^\beta\), \((1)_g^\beta\) and \((1)_p\), which are known functions of the zeroth order terms, as

\[
(1)_T^\beta = \left( (0)_m^{\alpha \gamma} (0)_T^\gamma \right)_{\alpha} - \left( (0)_m^{\alpha \gamma} (0)_W^{\gamma \beta} \right)_{\alpha} + \left( (0)_q^{\alpha} (0)_W^{\beta \alpha} \right)_{\alpha}
\]

\[
(1)_g^\beta = \left( (0)_m^{\alpha \gamma} (0)_T^\gamma \right)_{\beta} - \frac{1}{3} \left( (0)_m^{\alpha \gamma} (0)_W^{\gamma \beta} \right)_{\beta} - \left( (0)_q^{\alpha} (0)_W^{\beta \alpha} \right)_{\beta}
\]

\[
+ \left( (0)_q^{\alpha} (0)_W^{\beta \alpha} \right)_{\beta} + (0)_n^{33} (0)_W^{\beta \gamma} \]

(3.44)

If we eliminate \((k)_q^{\alpha}\) between (3.42)_2 and (3.42)_3 and substitute (3.40)_2 into the resulting expression we obtain the following set of equations, for \( k = 0, 1 \)

\[
(k)_H^{\alpha \beta} + (k)_T^\beta = 0
\]

\[-\frac{4}{3} \Delta_1 (1 + \Delta_0) (k)_W^{\alpha \beta} + \sum_{l=0}^{k} \left[ (0)_m^{\alpha \gamma} (k-\gamma)_W^{\gamma \beta} - (0)_T^\gamma (k-\gamma)_W^{\gamma \beta} \right] + (k)_n^{\alpha \beta} + (k)_g^\beta + (k)p = 0
\]

(3.45)

In a displacement type of formulation, these equations take the following form, using (3.35) and (3.40)_1, for \( k = 0, 1 \)

\[
(1 + 2\Delta_0) (k)_T^{\alpha \beta} + (k)_T^{\beta \alpha} + \sum_{l=0}^{k} \left[ (0)_m^{\alpha \gamma} (k-\gamma)_W^{\gamma \beta} + (1 + \Delta_0)(0)_W^{\alpha \beta} (k-\gamma)_W^{\gamma \beta} + \Delta_0 (0)_W^{\alpha \beta} (k-\gamma)_W^{\gamma \beta} \right]
\]

\[
+ \frac{1}{2\Delta_1} \left[ (k)_H^{\alpha \beta} + (k)_T^\beta \right] = 0
\]

\[-\frac{2}{3} (1 + \Delta_0) (k)_W^{\alpha \beta} + \sum_{l=0}^{k} \left[ (0)_T^{\alpha \gamma \beta \gamma} + (0)_T^{\alpha \gamma \beta \gamma} + \sum_{n=0}^{k} (0)_W^{\alpha \beta} (l-\gamma)_W^{\gamma \alpha} \right]
\]

\[
+ \Delta_0 \left[ (0)_T^{\alpha \gamma \beta \gamma} + \sum_{n=0}^{k} (0)_W^{\alpha \beta} (l-\gamma)_W^{\gamma \alpha} \right] (k-\gamma)_W^{\gamma \beta} \]

\[
+ \frac{1}{2\Delta_1} \left[ (0)_T^{\alpha \gamma \beta \gamma} (k-\gamma)_W^{\gamma \alpha} - (0)_T^{\alpha \gamma \beta \gamma} (k-\gamma)_W^{\gamma \alpha} + (k)_n^{\alpha \beta} + (k)_g^{\alpha \beta} + (k)_g^\alpha + (k)p \right] = 0
\]

(3.46)

where we have exactly three equations for three unknowns \((k)_T^{\alpha \beta}\) and \((k)_W^{\alpha \beta}\). In this case, the boundary conditions should of course be specified on \((k)_T^{\alpha \beta}\) and \((k)_W^{\alpha \beta}\) but they will be discussed later. It might be immediately noticed that these equations are nonlinear for \( k = 0 \) and are linear for \( k = 1 \), as is common in the asymptotic expansion technique. On the other hand, if the
boundary conditions are imposed on the surface tractions we choose \((k) \gamma^a\beta\) and \((k) \theta^a\beta\) as fundamental unknowns and consequently employ (3.45). But, in this case, we have only three equations for the four unknowns. On account of this, we need one more equation and this follows from the compatibility condition. Utilizing (3.35) we can easily obtain the following relation

\[
\varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta} (k) \gamma^a\gamma\beta - \sum_{i=0}^{k} \left( (0) \gamma_{i}^a, (k-i) \gamma_{i}^a \right) = \frac{\Delta_0}{1 + 2\Delta_0} \left( (k) \gamma^\gamma - (k) \theta^\gamma \right) \Delta_0 \alpha^\alpha \beta \tag{3.47}
\]

where \(\varepsilon_{\alpha\beta}\) is the two dimensional alternating tensor and we employ the notation

\[
[\Phi, \Psi] = \Phi_{\alpha}^\alpha \Phi_{\beta}^\beta - \Phi_{\alpha}^\beta \Phi_{\beta}^\alpha = \varepsilon_{\gamma\delta} \varepsilon^{\gamma\delta} \Phi_{\phi\gamma} \Phi_{\phi\delta} \Psi_{\alpha\beta}
\]

where \(\Phi\) and \(\Psi\) are scalar functions of \(\xi^1\) and \(\xi^2\). Moreover we can express \((k) \theta^a\beta\) from (3.40), as

\[
(k) \theta^a\beta = \frac{1}{2\Delta_0} \left[ \left( (k) \gamma^a\beta - (k) \theta^a\beta \right) \frac{\Delta_0}{1 + 2\Delta_0} \left( (k) \gamma^\gamma - (k) \theta^\gamma \right) \Delta_0 \alpha^\alpha \beta \right]
\]

Thus, substituting this result into (3.47) we obtain the compatibility condition in \((k) \gamma^a\beta\) and \((k) \theta^a\beta\) in the form

\[
(k) \gamma^\gamma - \frac{1 + \Delta_0}{1 + 2\Delta_0} (k) \gamma^\gamma = 2\Delta_0 \left[ \sum_{i=0}^{k} \left( (0) \gamma_{i}^a, (k-i) \gamma_{i}^a \right) + (k) \gamma^\gamma - \frac{1 + \Delta_0}{1 + 2\Delta_0} (k) \gamma^\gamma \right] \alpha^\alpha \beta \tag{3.48}
\]

To deduce the boundary conditions on \((k) \gamma^a\beta\) we integrate (3.9), and (3.10) with respect to \(\xi\) and then find that, for \(k = 0, 1\),

\[
(k) \gamma^a\beta_{|\xi^1} = (k) \beta^\beta \tag{3.49}
\]

where

\[
(0) \beta^\beta = \int_{\xi^1} \tau^\beta d\xi^1, \quad (0) \beta^\beta = - (0) \gamma^a\gamma(0) \beta^\beta - (0) \gamma^\gamma(0) \beta^\gamma - (0) \eta^\gamma(0) \beta^\gamma n^\gamma
\]

It is possible to express the system (3.45) and (3.48) by means of a stress function in a simpler form. To reach this form we assume that the solution (3.45) is in the form

\[
(k) \theta^a\beta = (k) \theta^a\beta + (k) \eta^a\beta
\]

where \((k) \theta^a\beta\) and \((k) \eta^a\beta\) satisfy the following relations

\[
(k) \theta^a\beta_{|\xi^1} = 0, \quad (k) \eta^a\beta_{|\xi^1} = - (k) \beta^\beta \tag{3.50}
\]

Now we can express \((k) \theta^a\beta\) as

\[
(k) \theta^a\beta = \varepsilon_{\gamma\delta} \varepsilon^{\gamma\delta} (k) \Phi_{\delta\gamma}
\]

which satisfies (3.50), identically. Here \((k) \Phi(\xi^1, \xi^2)\) is called the stress function. Using these definitions in (3.45) and (3.48) we obtain the following equations to determine \((k) \Phi\) and \((k) \theta\)

\[
(k) \Phi_{\gamma\gamma} = - \frac{1 + 2\Delta_0}{1 + \Delta_0} \left( 2\Delta_0 \sum_{i=0}^{k} \left( (0) \gamma_{i}^a, (k-i) \gamma_{i}^a \right) + (k) \gamma^\gamma - (k) \theta^\gamma \right) \alpha^\alpha \beta + (k) \eta^\gamma \alpha^\alpha \beta
\]

\[
- \frac{4}{3} \Delta_0 (1 + \Delta_0) (k) \theta_{\gamma\gamma} = \sum_{i=0}^{k} \left( (0) \eta^a\gamma(k-i) \theta_{\gamma\gamma}, i \alpha^\alpha \beta \right) \alpha = \sum_{i=0}^{k} (0) \Phi_{\gamma\gamma}, (k-i) \gamma_{i}^a \alpha^\alpha \beta + (k) \theta^\gamma \alpha^\alpha \beta + \sum_{i=0}^{k} (k) \eta^a\gamma, (k-i) \gamma_{i}^a \alpha^\alpha \beta
\]

These equations are none other than the well-known nonlinear von Kármán's equations in a somewhat generalized form for \(k = 0\). But they are linear for \(k = 1\). Employing (3.49) and (3.51) the boundary conditions on \((k) \Phi\) are found as follows

\[
\varepsilon_{\gamma\delta} \varepsilon^{\gamma\delta} (k) \Phi_{\delta\gamma} n^\alpha = (k) \theta^\beta - (k) \eta^a\beta n^\alpha \tag{3.51}
\]

When the applied tractions are given at the boundary, integrating \(\xi\) times (3.9) and (3.10), and (3.9) and (3.10) with respect to \(\xi\), we obtain the following boundary conditions

\[
(k) m^a\beta n^\alpha = (k) m^\beta, \quad (k) q^a\beta n^\alpha = (k) q^\beta - \sum_{i=0}^{k} (0) m^a(k-i) \gamma_{i}^a \alpha^\alpha \beta \tag{3.52}
\]

on \(c\)
where

\[ (0)m^\beta = \int_{-1}^{1} \tau^\beta \zeta \, d\zeta, \quad (0)h^3 = \int_{-1}^{1} \tau^3 \, d\zeta, \]

\[ (1)m^\beta = \frac{1}{3} (0)n_{\alpha \gamma} (0)W_{\gamma \beta} + (0)q^\alpha (0)W^\beta_{\gamma} - (0)m_{\alpha \gamma} (0)U_{\gamma \beta} n_{\alpha} \]

\[ (1)h^3 = \left\{ \Delta_0 (0)m_{\alpha \gamma} (0)U_{\gamma \beta} - \frac{1}{6} (0)n_{\alpha \gamma} (0)W_{\gamma \delta} \delta_{\beta \gamma} + (0)q^\alpha (0)U_{\gamma \beta} - (0)q^\alpha (0)W_{\gamma \gamma} \right\} n_{\alpha} + \frac{1}{2} (1 + \Delta_0) \left\{ (0)q^\alpha (0)W_{\gamma \gamma} (0)W_{\gamma \gamma} + (0)m_{\alpha \gamma} (0)W_{\gamma \gamma} (0)W_{\gamma \gamma} \right\} n_{\alpha} \]

Using (3.42) and (3.40)2 we obtain \((k)q^\beta\) in the form

\[ (k)q^\beta = -\frac{4}{3} \Delta_0 (1 + \Delta_0) \left( (k)W_{\beta \alpha} + (k)\tilde{n}_{\alpha \beta} + (k)g^\beta \right) \]

Since the three conditions (3.52) are more than the necessary conditions to determine \((k)W\) uniquely, as in the classical plate theory the boundary conditions are written on the bending moment \((k)m_n = (k)m_{\alpha \beta} n_{\alpha} n_{\beta}\) and the effective shearing force

\[ (k)q_s = \frac{\partial (k)m_s}{\partial s} \quad \text{on } c \]

Here, \((k)m_n = (k)m_{\alpha \beta} n_{\alpha} n_{\beta}\) is the twisting moment, \(t\) is the unit tangent vector to the boundary curve and \(s\) is the arc length along the boundary. Then the new boundary conditions are given by

\[ (k)m_n = (k)m_{\alpha \beta} n_{\alpha} n_{\beta}, \quad (k)q_s = \frac{\partial (k)m_s}{\partial s} = \frac{\partial}{\partial s} \left( (k)m_{\alpha \beta} n_{\alpha} \right) \quad \text{on } c \]

As we mentioned above, to determine \((1)U_{\beta \alpha}, (1)\tilde{n}_{\alpha \beta}, (1)\tilde{n}_{\alpha \beta}\) and \((1)\tilde{n}_{\alpha \beta}\) we need to know \((0)\sigma^{33}\) and \((0)\sigma^{33}\). Therefore to compute \((0)\sigma^{33}\) we transform (3.7), into

\[ (0)\sigma^{33}_3 = -\frac{3}{2} \xi (0)m_{\alpha \beta} + \frac{1}{2} (0)r_{\beta} - f_{\beta} \]

where \(m_{\alpha \beta}\) and (3.41) are arbitrary functions of \(\xi^1, \xi^2\). To determine \(F_{\beta}\) we substitute (3.53) into the boundary conditions (3.9), and then we obtain two equations. While subtraction of these two equations leads to an identity, their addition yields

\[ F_{\beta}(\xi^1, \xi^2) = \frac{3}{4} (0)m_{\alpha \beta} + \frac{1}{2} (0)i_{\beta} \]

where

\[ (0)i_{\beta} = j^\alpha_{\beta} = \zeta^\alpha_{\beta} + \int_{-1}^{1} f_{\beta} \, d\zeta \]

Thus, using (3.41) and (3.40), we obtain \((0)\sigma^{33}\) as follows

\[ (0)\sigma^{33} = -\Delta_0 (1 + \Delta_0) (1 - \xi^2) (0)m_{\alpha \beta} + \frac{1}{2} \xi (0)r_{\beta} + \frac{1}{2} (0)i_{\beta} - \int_{-1}^{1} f_{\beta} \, d\zeta \]

Similarly, to determine \((0)\sigma^{33}\) we can write (3.7) in the form

\[ (0)\sigma^{33}_3 = - (0)\sigma^{33}_3 - (0)\sigma^{33}_3 + f_{\beta} (0)W_{\gamma \alpha} - f_{3} \]

where we used (3.7). Employing (3.55) and (3.41) in this equation and integrating the result
with respect to $\xi$, we get the following relation

$$ (0)\sigma^{33} = \Delta_{1}(1 + \Delta_{0})\xi \left(1 - \frac{1}{3}\xi^{2}\right) (0)\omega^{\alpha}_{\beta} - \frac{3}{4}\xi^{2} (0)m^{a\beta} (0)b^{a\beta} \right) \\
+ \frac{1}{4}\xi^{2} (0)\omega^{a\beta}_{\alpha} - \frac{1}{2} (0)i^{a\beta}_{\alpha} + \xi \int_{-1}^{1} f^{a\beta}_{\alpha} d\xi \\
- \int_{-1}^{1} f^{a\beta}_{\alpha} d\xi + (0)\omega^{a\beta}_{\alpha} \int_{-1}^{1} f^{a\beta}_{\alpha} d\xi - \int_{-1}^{1} f^{a\beta}_{\alpha} d\xi + F(\xi^{1}, \xi^{2}) \right) (3.56) $$

where $F(\xi^{1}, \xi^{2})$ is an arbitrary function of $\xi^{1}$ and $\xi^{2}$. The use of this equation in the boundary condition (3.9)\textsubscript{2} which now is written in the form

$$(0)\sigma^{33} = \mp (g_{+}^{3} - g_{-}^{3}, (0)\omega_{\alpha}) \quad \text{on } \omega^{+}$$

gives two equations whose subtraction from each other leads to an identity, again. However, by adding them we find that

$$ F(\xi^{1}, \xi^{2}) = \frac{3}{4}(0)m^{a\beta} (0)\omega^{a\beta}_{\alpha} + \frac{1}{4}(0)i^{a\beta}_{\alpha} + \frac{1}{2}(0)i^{a\beta}_{\alpha} - \frac{1}{2}(0)i^{a\beta}_{\alpha} + \frac{1}{2}(0)m^{a\beta} (0)\omega^{a\beta}_{\alpha} \right) (3.57) $$

where

$$ (0)i = g_{+}^{3} - g_{-}^{3} + \int_{-1}^{1} f^{a\beta}_{\alpha} d\xi \right) (3.58) $$

Now, utilizing (3.57), (3.40)\textsubscript{2} and (3.45)\textsubscript{2} in (3.56), we obtain $(0)\sigma^{33}$ as

$$(0)\sigma^{33} = \frac{1}{3}\Delta_{1}(1 + \Delta_{0})(\xi - \xi^{2}) (0)\omega^{a\beta}_{\alpha} = \Delta_{1}(1 - \xi^{2})(0)m^{a\beta} (0)\omega^{a\beta}_{\alpha} \right) \\
+ \Delta_{0}(0)\omega^{a\beta}_{\alpha} (0)\omega^{a\beta}_{\alpha} - \frac{1}{2}(0)\omega^{a\beta}_{\alpha} (0)i^{a\beta}_{\alpha} + \xi \int_{-1}^{1} f^{a\beta}_{\alpha} d\xi \\
+ \frac{1}{2}(0)i^{a\beta}_{\alpha} + 2 \int_{-1}^{1} f^{a\beta}_{\alpha} d\xi + \frac{1}{2}(1 + \xi)(0)s^{a\beta}_{\alpha} - (0)i^{a\beta}_{\alpha} + \frac{1}{4}(1 - \xi^{2}) \right) \times (0)\omega^{a\beta}_{\alpha} + \int_{-1}^{1} (\xi - \eta)f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta \right) $$

We see now that to determine $(0)\sigma^{33}$ and $(0)\sigma^{33}$, it suffices to know only $(0)\omega$ which is a solution of the zeroth order problem. Now, employing the definitions (3.39) we obtain $(0)\sigma^{a\beta}_{\alpha}$, $(0)m^{33}$ and $(0)m^{33}$ in the form

$$(0)\sigma^{a\beta}_{\alpha} = \frac{1}{3}(0)i^{a\beta}_{\alpha} - \int_{-1}^{1} \xi \int_{-1}^{1} f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi \right)$$

$$(0)m^{33} = \frac{1}{3}\Delta_{1}(1 + \Delta_{0})(0)\omega^{a\beta}_{\alpha} = \Delta_{1}(0)\omega^{a\beta}_{\alpha} (0)\omega^{a\beta}_{\alpha} + \Delta_{0}(0)\omega^{a\beta}_{\alpha} (0)\omega^{a\beta}_{\alpha} - (0)i^{a\beta}_{\alpha} (0)\omega^{a\beta}_{\alpha} + (0)i^{a\beta}_{\alpha} + (0)m^{33} \right) \\
- \int_{-1}^{1} \int_{-1}^{1} f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi - \int_{-1}^{1} \int_{-1}^{1} f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi \right) + \int_{-1}^{1} \int_{-1}^{1} (\xi - \eta)f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi \right) \right)$$

$$(0)s^{a\beta}_{\alpha} = \frac{1}{3}(0)i^{a\beta}_{\alpha} - \int_{-1}^{1} \int_{-1}^{1} f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi - \int_{-1}^{1} \int_{-1}^{1} f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi \right) \right)$$

$$(0)m^{33} = \frac{1}{3}\Delta_{1}(1 + \Delta_{0})(0)\omega^{a\beta}_{\alpha} = \Delta_{1}(0)\omega^{a\beta}_{\alpha} (0)\omega^{a\beta}_{\alpha} + \Delta_{0}(0)\omega^{a\beta}_{\alpha} (0)\omega^{a\beta}_{\alpha} - (0)i^{a\beta}_{\alpha} (0)\omega^{a\beta}_{\alpha} + (0)i^{a\beta}_{\alpha} + (0)s^{a\beta}_{\alpha} \right) \\
- \int_{-1}^{1} \int_{-1}^{1} f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi - \int_{-1}^{1} \int_{-1}^{1} f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi \right) + \int_{-1}^{1} \int_{-1}^{1} (\xi - \eta)f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi \right) \right)$$

$$(0)m^{33} = \frac{1}{3}\Delta_{1}(1 + \Delta_{0})(0)\omega^{a\beta}_{\alpha} = \Delta_{1}(0)\omega^{a\beta}_{\alpha} (0)\omega^{a\beta}_{\alpha} + \Delta_{0}(0)\omega^{a\beta}_{\alpha} (0)\omega^{a\beta}_{\alpha} - (0)i^{a\beta}_{\alpha} (0)\omega^{a\beta}_{\alpha} + (0)i^{a\beta}_{\alpha} + (0)m^{33} \right) \\
- \int_{-1}^{1} \int_{-1}^{1} f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi - \int_{-1}^{1} \int_{-1}^{1} f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi \right) + \int_{-1}^{1} \int_{-1}^{1} (\xi - \eta)f^{a\beta}_{\alpha}(\xi^{1}, \eta) d\eta d\xi \right) \right)$$
It is now in order to remark that although in the zeroth order (von Kármán's) theory partition of surface loads between upper and lower surfaces of the plate is immaterial and deflections only change sign when the resultant load changes from pressure to tension or vice versa keeping its magnitude constant, the situation is entirely different in the first order theory where we have to specify precisely how the load is applied. To see this more clearly let us assume that $g^z = (0) = f^z = f^3 = 0$ or equivalently $(0)^z = (0)^z = 0$. Let us first compare the pair of equations $g^z = p_0$ and $g^z = 0$ with the pair $g^z = \frac{1}{2} p_0$ and $g^z = \frac{1}{2} p_0$ where $p_0$ is a uniform load. Employing (3.43) and (3.59), we see that $(0)p = p_0$ in both cases and that $(0)I = p_0$ and $(0)I = 0$, respectively. Therefore, the stress components $(0)\sigma^{33}$ corresponding to these two cases differ from each other by $\frac{1}{2} p_0$. Let us then compare $g^z = p_0$ and $g^z = 0$ with $g^z = -p_0$ and $g^z = 0$. In this case, we find that $(0)p = p_0$ and $(0)I = -p_0$, respectively. Noting that $\zeta$ and $(0)\psi$ have opposite signs in both cases, we see that the absolute values of $(0)\sigma^{33}$ corresponding to these two cases differ from each other by $p_0$. The same is also true for $(0)\eta^{33}$. Moreover, this difference affects the functions $(1)\sigma^{ab}$, $(1)\tau^{ab}$, $(1)\eta^{ab}$ and $(1)\beta^{ab}$ and consequently the right hand side of (3.46) for $k = 1$. Therefore, it is important to know how the load is applied in the first order approximation. Obviously the gist of the matter is that the transverse stress components which are neglected in the von Kármán theory are effective in the first order approximation.

Since we already know $(0)\sigma^{30}$ and $(0)\sigma^{33}$ we can now calculate the functions $(1)\sigma^{ab}$, $(1)\tau^{ab}$, $(1)\eta^{ab}$ and $(1)\beta^{ab}$ given by (3.34), (3.36), (3.38) and (3.40) respectively. Thus, introducing (3.55) into (3.33) and integrating the resulting expression with respect to $\zeta$ we find that

\[
(1)\sigma^{ab} = \left[ (1 + \Delta_0)\xi^3 - \frac{1}{6} (2 + \Delta_0)\xi^3 \right] (0)\psi;_{\delta}^{\sigma \beta} + \frac{1}{4} \Delta_0 \psi^3 (0)\theta;_{\delta}^{\sigma \beta} - 2 (0)\psi;_{\delta}^{\sigma \beta} (0)\psi;_{\delta}^{\sigma \beta}
\]

Substituting this result into (3.37) we get $(1)\theta^{ab}$ as

\[
(1)\theta^{ab} = \left[ (1 + \Delta_0)\xi^3 - \frac{1}{6} (2 + \Delta_0)\xi^3 \right] (0)\theta;_{\delta}^{\sigma \beta} + \frac{1}{4} \Delta_0 (0)\theta;_{\delta}^{\sigma \beta} - 2 (0)\theta;_{\delta}^{\sigma \beta} (0)\psi;_{\delta}^{\sigma \beta}
\]

To obtain $(1)\tau^{ab}$ we must substitute $(1)\theta^{ab}$ instead of $(1)\psi^{ab}$, in (3.22)$_2$. Then, after some calculations we obtain $(1)\eta^{ab}$ in the form

\[
(1)\eta^{ab} = \frac{1}{2} (0)\eta^{ab} + \frac{1}{2} (0)\eta^{ab} + \frac{1}{2} (0)\eta^{ab}
\]

where $(0)\eta^{ab}$ ($n = 0, 1, 2, 3$) are given in the Appendix A. Similarly, employing this relation in the
definitions (3.39) \( R_{\alpha\beta}^{\rho\sigma} \) and \( \tilde{R}_{\alpha\beta}^{\rho\sigma} \) are found as

\[
(1) R_{\alpha\beta}^{\rho\sigma} = I_0^{\rho\sigma} + \frac{1}{3} I_2^{\rho\sigma} - \Delta_0 \left[ \int_{-1}^{1} \left( \xi - \eta \right) f_{\gamma}^{\rho} \, d\eta \, d\xi \right] A_{\alpha\beta}^{\rho\sigma} - \int_{-1}^{1} \int_{-1}^{1} \left( \xi - \eta \right) \left( f_{\alpha}^{\rho\sigma} + f_{\beta}^{\rho\sigma} \right) \, d\eta \, d\xi
\]

\[
(1) \tilde{R}_{\alpha\beta}^{\rho\sigma} = \frac{1}{3} I_1^{\rho\sigma} + \frac{1}{5} I_3^{\rho\sigma} - \Delta_0 \left[ \int_{-1}^{1} \xi \int_{-1}^{1} \left( \xi - \eta \right) f_{\gamma}^{\rho} \, d\eta \, d\xi \right] A_{\alpha\beta}^{\rho\sigma} - \int_{-1}^{1} \int_{-1}^{1} \left( \xi - \eta \right) \left( f_{\alpha}^{\rho\sigma} + f_{\beta}^{\rho\sigma} \right) \, d\eta \, d\xi
\]

(3.61)

We know that, in the zeroth order approximation, the boundary conditions will be prescribed by \( \vec{u}_\alpha \) and \( \vec{w} \). However, in the first order approximation, since \( \vec{u}_k \ (k = 1, 2, 3) \) are also functions of \( \xi \) the integration constants are less than those required by boundary conditions. To remedy this situation two artifices may be used. One way of approach is to prescribe the boundary conditions on averaged displacements defined by

\[
(1) \bar{u}_k(\xi^1, \xi^2) = \frac{1}{2} \int_{-1}^{1} \bar{u}_k(\xi^1, \xi^2, \xi) \, d\xi, \quad k = 1, 2, 3
\]

Employing (3.31), (3.32), (3.34) and (3.59) we obtain the averaged displacements as follows

\[
(1) \bar{u}_\alpha = \left( \vec{u}_\alpha \right) + \frac{1}{12} \Delta_0 \left( \theta_{\beta,\alpha}^{\rho\sigma} - 2 \theta_{\beta,\sigma}^{\rho\rho} \bar{w}_{\alpha} \right) + \frac{1}{12} \bar{w}_{\alpha} \left( \frac{1}{2} \Delta_1 \right) \int_{-1}^{1} \int_{-1}^{1} \left( \xi - \eta \right) f_{\alpha}^{\rho\sigma} \, d\eta \, d\xi
\]

\[
(1) \bar{u}_3 = \left( \vec{u}_3 \right) + \frac{1}{6} \Delta_0 \left( \theta_{\beta,\sigma}^{\rho\rho} \right)
\]

Since the approximation presented here should be considered as an outer solution to the actual boundary layer problem, the above assumption seems reasonable. The other way is to prescribe the boundary conditions on the middle plane only, that is, for \( \xi = 0 \). If the body forces are not taken into account, then \( \bar{u}_\alpha = \bar{u}_k = 0 \) for \( \xi = 0 \). Thus this is equivalent to write the boundary conditions on \( \bar{w} \) and \( \bar{u}_\alpha \).

In Appendix B, the explicit forms of governing equations of the zeroth and first order approximations for different formulations are summarized.

Finally we would like to find the Cauchy stress tensor which represents the actual state of stress in the deformed plate. To this end we use the relation

\[\bar{t}^{k\ell} = J^{-1} \bar{x}^k_{J,\ell} \bar{T}^{k\ell} \]

(3.62)

where \( J = III_{J}^{\ell} \). Using (3.25), (3.5), (3.6), (3.2) and the relation \( J = III_{J}^{\ell} \) we obtain the expansion of \( J^{-1} \) in the form

\[
J^{-1} = 1 - (1 - \Delta_0) \left( \vec{u}_3^{\rho\sigma} + \frac{1}{2} \vec{u}_{3;\rho\sigma} \right) \varepsilon^2 + O(\varepsilon^4)
\]

(3.63)

Thus, employing (2.4), (2.12) and (3.63) in (3.62) we get after some lengthy calculations

\[
\frac{1}{I_0} t^{k\ell} = \varepsilon^2 \left( \sigma^{\rho\sigma} \delta_{\delta\beta}^{\rho\sigma} + \varepsilon^2 \left[ (1) \sigma^{\rho\sigma} + (0) \sigma^{\alpha\alpha} (0) u_{\gamma}^{\rho} + (0) \sigma^{\gamma\beta} (0) u_{\gamma}^{\rho} \right. \right. \\
(0) \sigma^{\rho\sigma} (0) u_{\gamma}^{\rho} + \left. \left. (0) \sigma^{\rho\sigma} (0) u_{\gamma}^{\rho} \right] \delta_{\delta\beta}^{\rho\sigma} + O(\varepsilon^4) \right)
\]
\[
\frac{1}{T_0} \varepsilon^3 = \varepsilon^3 \left[ \sigma^{\alpha \alpha} + (0) \sigma^{\alpha \beta} (0) u_{\beta}^3 \right] g_\alpha^\beta + \varepsilon^3 \left[ (0) \sigma^{\alpha \alpha} + (1) \sigma^{\alpha \beta} (0) u_{\beta}^3 + (0) \sigma^{\alpha \beta} (1) u_{\beta}^3 \right] \\
+ (0) \sigma^{\alpha \beta} (1) u_{\beta}^3 + (0) \sigma^{\beta \alpha} (0) u_{\alpha}^3 + (0) \sigma^{\alpha \beta} (0) u_{\alpha}^3 + (0) \sigma^{\beta \alpha} (0) u_{\beta}^3 \\
+ (0) \sigma^{\beta \alpha} (0) u_{\beta}^3 (0) u_{\alpha}^3 - (1 - \Delta \theta) \left( (0) u_{\alpha}^3 + \frac{1}{2} (0) u_{\beta}^3 (0) u_{\alpha}^3 \right) \\
\times (0) \sigma^{\alpha \alpha} + (0) \sigma^{\alpha \beta} (0) u_{\beta}^3 \right] g_\alpha^\beta + O(\varepsilon^5)
\]

\[
\frac{1}{T_0} \varepsilon^3 = \varepsilon^3 \left( (0) \sigma^{33} + 2 (0) \sigma^{3 \alpha} (0) u_{\alpha}^3 + (0) \sigma^{\alpha \beta} (0) u_{\alpha}^3 (0) u_{\beta}^3 \right) + \varepsilon^3 \left[ (0) \sigma^{33} \\
+ 2 (0) \sigma^{3 \alpha} (1) u_{\alpha}^3 + 2 (1) \sigma^{3 \alpha} (0) u_{\alpha}^3 + 2 (0) \sigma^{33} (1) u_{3}^3 + 2 (0) \sigma^{\alpha \beta} (0) u_{\alpha}^3 (1) u_{\beta}^3 \\
+ (0) \sigma^{\alpha \beta} (0) u_{\alpha}^3 (0) u_{\alpha}^3 + (0) \sigma^{\alpha \beta} (0) u_{\alpha}^3 (0) u_{\beta}^3 - (1 - \Delta \theta) \left( (0) u_{\alpha}^3 \right) \\
\times (0) \sigma^{3 \alpha} + (0) \sigma^{3 \beta} (0) u_{\beta}^3 \right] g_\alpha^\beta + O(\varepsilon^5)
\]

where \( a \) and \( b \) take values 1 and 2 only. Let us recall that \( g_\alpha^\beta = g_\alpha^\beta \cdot G_\alpha \) are shifters.

4. CONCLUSIONS

In this work, an asymptotic theory for thin, compressible hyperelastic plates is derived. The first two order field equations are obtained and it is shown that the zeroth order theory corresponds to the well-known von Kármán theory which is tantamount to state that physical nonlinearity becomes effective only at the first order approximation. On the other hand, in the first order approximation it has been observed that the total stress cannot be decomposed into bending and membrane stress components. Furthermore the stress components in the transverse direction have been determined and the Cauchy stress tensor is also evaluated.

We would like to emphasize the fact that our aim here is to develop a systematic theory which can relatively easily be applied to engineering problems. We do not intend to provide a rigorous mathematical analysis which would take care more naturally convergence, existence and uniqueness questions. That is the reason why we opted to work directly with field equations rather than resorting to a variational approach which has the firm support of functional analysis.

One feature of the present analysis is that the effect of material nonlinearity becomes apparent only in a later stage of the asymptotic expansion. We expect that, for highly nonlinear materials, this can be remedied by appropriately modifying the scaling of the field quantities. Similarly, it is possible to study the edge effects, in other words, the inner solution in the boundary layer region by rescaling the spatial coordinates. These will be subjects for further studies.

We believe that our analysis might have applications in designing elastomers which has an ever-increasing use in technology. However, since many elastomers are incompressible it is advantageous to have a theory directly applicable to nonlinear elastic incompressible materials. This will be investigated in the sequel.

REFERENCES

APPENDIX A

The explicit representations of the tensors $I_{\alpha\beta}^n (n = 0, 1, 2, 3)$ in (3.60) and (3.61)

\[
I_{\alpha\beta}^0 = 2\Delta_0 \Delta_1 \left[ (0)_{\gamma \gamma} \cdot (0)_{\gamma \gamma} - \Delta_0 (0)_{\gamma \gamma} \cdot (0)_{\gamma \gamma} \right] + \frac{1}{4} \Delta_0 \Delta_1 \left[ (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} \right] + \frac{1}{2} \Delta_0 \Delta_1 (0)_{\gamma \gamma} (0)_{\gamma \gamma} + \Delta_0 (0)_{\gamma \gamma}
\]

\[
I_{\alpha\beta}^1 = \left[ -\frac{10}{3} \Delta_0 \Delta_1 (1 + \Delta_0) (0)_{\gamma \gamma} (0)_{\gamma \gamma} + 2 \Delta_0 \Delta_1 (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma}ight]
\]

\[
I_{\alpha\beta}^2 = \Delta_0 \Delta_1 (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} + 2 \Delta_0 \Delta_1 (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma}
\]

\[
I_{\alpha\beta}^3 = \frac{2}{3} \Delta_0 (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (2 + \Delta_0) (0)_{\gamma \gamma} (0)_{\gamma \gamma}
\]

APPENDIX B

The basic equations of the zeroth and first order theory

B1. The Zeroth Order Approximation

B1.1 The plate equations

In a displacement type of formulation;

\[
(1 + 2\Delta_0) (0)_{\gamma \gamma} + (0)_{\gamma \gamma} + (0)_{\gamma \gamma} = 0
\]

\[
\frac{4}{3} \Delta_1 (1 + \Delta_0) (0)_{\gamma \gamma} (0)_{\gamma \gamma} - 2 \Delta_1 (0)_{\gamma \gamma} (0)_{\gamma \gamma} + \Delta_0 (2) (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} = \frac{2}{3} (0)_{\gamma \gamma} + \frac{1}{2} \Delta_1 (0)_{\gamma \gamma} (0)_{\gamma \gamma}
\]

In a stress type of formulation;

\[
(0)_{\gamma \gamma} = 0
\]

\[
\frac{4}{3} \Delta_1 (1 + \Delta_0) (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} = \frac{1}{2} \Delta_1 (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} + (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} = \frac{1}{2} (0)_{\gamma \gamma} + \frac{1}{2} \Delta_1 (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma} (0)_{\gamma \gamma}
\]
or

\[ 4 \Delta_1 (1 + \Delta_0) \Phi_\alpha \sigma \beta - \frac{1 + 2 \Delta_0}{1 + \Delta_0} 2 \Delta_1 (W_\alpha \sigma \beta) \Phi_\alpha \sigma \beta = \frac{1 + 2 \Delta_0}{1 + \Delta_0} \Phi_\alpha \sigma \beta - \frac{1 + 2 \Delta_0}{1 + \Delta_0} \Phi_\alpha \sigma \beta \]

\[ \frac{4}{3} \Delta_1 (1 + \Delta_0) W_\alpha \sigma \beta - \frac{1 + 2 \Delta_0}{1 + \Delta_0} \Phi_\alpha \sigma \beta = \left[ (0) W_\alpha \right] = \left[ (0) W_\alpha \right] + \left[ (0) p_\alpha \right] + (0) p \]

\[ B1.2 \text{ The boundary conditions} \]

The in-plane boundary conditions are given by

\[ (0) v_\sigma = 0 \quad \text{on } c \]

for a displacement type of formulation and by

\[ (0) n_\alpha n_\beta = (0) n_\beta \quad \text{or} \quad \varepsilon_{\alpha \beta} e^{\alpha \beta \gamma} (0) \Phi_{\gamma \sigma \beta} n_\sigma = (0) H_{\alpha \beta} \quad \text{on } c \]

for a stress type of formulation. On the other hand, some boundary conditions related to transverse displacement component are specified by

(a) Clamped edge: \[ (0) w = 0, \quad \frac{\partial (0) w}{\partial n} = 0 \quad \text{on } c_1 \]

(b) Simply supported edge: \[ (0) w = 0, \quad (0) m_\alpha = (0) m_\alpha n_\beta \quad \text{on } c_2 \]

(c) Free edge: \[ (0) q_\alpha = (0) m_\alpha, \quad \frac{\partial (0) q_\alpha}{\partial n} = (0) n_\alpha \beta (0) w_\beta + \frac{3}{2} (0) m_\alpha \beta \quad \text{on } c_3 \]

where \( c_1, c_2 \) and \( c_3 \) denote clamped, simply supported and free parts of \( c \).

\[ B2. \text{ The First Order Approximation} \]

\[ B2.1 \text{ The plate equations} \]

In a displacement type of formulation;

\[ (1 + 2 \Delta_0) (1) v_\alpha \sigma \beta + (1) v_\beta \sigma \alpha + (1 + 2 \Delta_0) (0) W_\alpha \sigma \beta + (1) W_\alpha \sigma \beta + (1) W_\alpha \sigma \beta + (0) W_\alpha \sigma \beta + (0) \Phi_\alpha \sigma \beta = \frac{1}{2 \Delta_1} (1) e^{\alpha \sigma \beta} + (0) p \]

\[ \frac{4}{3} \Delta_1 (1 + \Delta_0) (1) W_\alpha \sigma \beta - 2 \Delta_1 (0) V_\alpha \beta + (0) \Phi_{\gamma \sigma \beta} (1) W_\alpha \sigma \beta + (0) \Phi_{\gamma \sigma \beta} + (1) V_{\alpha \beta} + (0) \Phi_{\gamma \sigma \beta} (0) W_\sigma \beta \]

\[ + \Delta_0 (2) W_\alpha \beta + (0) W_\alpha \beta (0) W_\beta \sigma (0) W_\rho \sigma (0) W_\rho \sigma (0) W_\rho \sigma (0) W_\rho \sigma (1) V_{\alpha \beta} + (1) V_{\alpha \beta} (0) W_\sigma \beta \]

\[ + (0) W_\sigma \beta (0) W_\rho \sigma (0) W_\rho \sigma (0) W_\rho \sigma (0) W_\rho \sigma (0) W_\rho \sigma (0) W_\rho \sigma (0) W_\rho \sigma (0) W_\rho \sigma (1) p \]

\[ + (1) \Phi_\alpha \sigma \beta = (0) \Phi_\alpha \sigma \beta + (0) \Phi_\alpha \sigma \beta + (1) \Phi_\alpha \sigma \beta + (0) \Phi_\alpha \sigma \beta \]

In a stress type of formulation;

\[ (1) n_\sigma \sigma \beta + (1) \Phi_\alpha \sigma \beta = 0 \]

\[ \frac{4}{3} \Delta_1 (1 + \Delta_0) (1) W_\alpha \sigma \beta - (0) n_\alpha \beta (0) W_\beta \sigma + (0) e^{\alpha \sigma \beta} (0) \Phi_{\gamma \sigma \beta} + (0) \Phi_{\gamma \sigma \beta} (1) W_\alpha \sigma \beta = (1) p + (1) \Phi_\alpha \sigma \beta + (0) \Phi_{\gamma \sigma \beta} - (1) \Phi_\alpha \sigma \beta \]

\[ (1) n_\sigma \sigma \beta + (1) \Phi_\alpha \sigma \beta = \frac{1 + \Delta_0}{1 + \Delta_0} (0) n_\alpha \beta \quad \text{on } c \]

or

\[ (0) \Phi_\alpha \sigma \beta = \frac{1 + 2 \Delta_0}{1 + \Delta_0} (4 \Delta_1 (0) W_\alpha \beta + (1) e^{\alpha \beta \gamma} + (1) \Phi_{\gamma \sigma \beta} + (0) \Phi_\alpha \beta + (1) \Phi_\alpha \beta + (0) \Phi_{\gamma \sigma \beta} (1) p + (1) \Phi_\alpha \beta + (1) \Phi_{\gamma \sigma \beta} + (0) \Phi_{\gamma \sigma \beta} (0) W_\beta \sigma) \]

\[ B2.2 \text{ The boundary conditions} \]

Here we give the boundary conditions imposed on the averaged displacements. However, the boundary conditions written on the middle plane are also presented in parentheses. As in the zeroth order approximation, the in-plane boundary conditions are given in the form

\[ (1) u_\alpha = 0 \quad (0) u_\alpha = 0 \quad \text{on } c \]
for a displacement type of formulation and in the form

\[(1) \sigma^m_{\theta \theta} n_\theta = (1) \sigma^m_{\phi \phi} \]  

\[= \varepsilon^m_{\theta \theta} \Phi_{\theta \theta} n_\theta = (1) \sigma^m_{\phi \phi} \quad \text{on } c\]

for a stress type of formulation. The boundary conditions related to the transverse direction are presented as follows

(a) Clamped edge: \( (1) \dot{u}_3 = 0, \quad \frac{\partial (1) \ddot{u}_3}{\partial n} = 0 \left( \text{or } (1) w = 0, \frac{\partial (1) w}{\partial n} = 0 \right) \quad \text{on } c_1 \)

(b) Simply supported edge: \( (1) \dot{u}_3 = 0 \) (or \( (1) w = 0 \)), \( (1) m_\theta = (1) m_\phi \quad \text{on } c_2 \)

(c) Free edge: \( (1) m_\theta = (1) m_\phi \),

\[ (1) \ddot{q}_x = (1) \ddot{q}_m - \frac{\partial (1) m_\phi}{\partial x} = (1) \Phi_{\phi \phi} \left( (1) w_{,x} - (1) \dot{u}_x \right) \quad \text{on } c_3 \]