NONLINEAR WAVE MODULATION IN MICROPOLAR ELASTIC MEDIA—II. TRANSVERSE WAVES†

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Abstract—Nonlinear modulation of transverse waves in an infinite micropolar elastic medium is investigated under the assumption of weak nonlinearity. Using the reductive perturbation method, it is shown that the slowly varying complex amplitudes of the transverse (displacement or microrotation) waves are governed by two coupled Nonlinear Schrödinger (NLS) equations. Some special solutions of these coupled equations, namely, circularly and linearly polarized nonlinear plane wave and envelope solitary wave solutions, are given. The modulational instability of the plane wave solutions is also discussed.

1. INTRODUCTION

In the first part of this study [1], the nonlinear amplitude modulation of the longitudinal plane waves propagating in a weakly nonlinear micropolar elastic medium was examined using the reductive perturbation method. The longitudinal microrotation wave was the only dispersive longitudinal wave in the medium, and it was shown that the slowly varying complex amplitude of this wave is governed by the single Nonlinear Schrödinger (NLS) equation. When there exist various dispersive waves in a medium, under certain conditions, a variety of interesting phenomena can arise as a result of a simultaneous action of dispersion and nonlinearity giving rise to self- and mutual interactions. For instance, in a linear isotropic micropolar elastic medium, the dispersion relation associated with the pair consisting of a transverse displacement component in one direction and a transverse microrotation component in the other direction is identical to that corresponding to the other pair of transverse components. This property of linear transverse plane waves, i.e. possession of an identical dispersion relation, may lead to a nonlinear coupling of these waves if nonlinearity is included in the medium. Because of this feature of linear transverse waves, in the long-wave approximation, two coupled Modified Korteweg–de Vries equations were obtained for the propagation of plane waves in nonlinear micropolar elastic solids in [2]. This feature of transverse waves also constitutes the crucial point of the present analysis.

In this paper, we extend the study of nonlinear wave modulation to transverse waves propagating in a weakly nonlinear micropolar elastic medium. The method is the same: the reductive perturbation method. In Section 2, following the usual procedure, we first find that in the first-order approximation all the first-order longitudinal wave amplitudes are zero whereas the corresponding transverse wave amplitudes are not zero. In addition, all the first-order transverse wave amplitudes are expressed in terms of the amplitudes of two linearly polarized transverse displacement waves. We then investigate how these two independent waves interact each other as a result of weak nonlinearity. As a result, we show that the complex amplitudes of two linearly polarized transverse displacement waves are governed by two coupled nonlinear equations including the usual NLS equation as a special case. Similar equations are also valid with slightly different coefficients for the amplitudes of transverse microrotation waves. Then rewriting the system of these coupled nonlinear equations in terms of the amplitudes of two circularly polarized waves, we show that these equations reduce to two coupled NLS equations. In the first subsection, the solutions corresponding to some special cases are presented, and the corresponding displacement gradient components are given in terms of the original variables. These solutions include circularly and linearly polarized waves with "sech envelope" and

†The results presented here were obtained in the course of research sponsored by the Natural Science and Engineering Research Council of Canada, Grant No. OGP99118.
nonlinear wavetrains. Finally, in the second subsection, the modulation instability of the plane wave solutions is examined and it is observed that, in general, the coupling of these waves tends to cause the instability. Section 3 is devoted to concluding remarks.

2. MODULATION OF TRANSVERSE WAVES

For the modulation of transverse waves, we now assume \( D_3(k, \omega) = 0 \) but \( D_1(k, \omega) \neq 0 \) and \( D_2(k, \omega) \neq 0 \) for the pair \((k, \omega)\), which is contrary to what we have assumed in Part I. The explicit forms of functions \( D_1, D_2 \) and \( D_3 \) were given in equations (2.22, 1), where I refer to Part I. In the case of transverse waves, an interesting situation arises, which is different from that for longitudinal waves. That is, transverse waves corresponding to pairs \((u_3, \varphi_3)\) and \((u_3, \varphi_3)\) have the same dispersion relation, i.e. \( D_3(k, \omega) = 0 \), and consequently the same group velocity, given by

\[
c_s = \frac{k[2\omega^2(c_s^3 + b_s^3) - 4b_s^3c_s^3k^2 - 2c_s^3\omega_0^2 + \omega_0^2c_s^3]}{2\omega[2\omega^2 - (c_s^3 + b_s^3)k^2 - \omega_0^2]} \tag{2.1}
\]

We may therefore expect wave–wave interactions between these transverse waves.

Using the stretched coordinates (3.3, 1) and the expansions (3.4, 1) in the field equations (3.2, 1) we obtain successive systems of equations for each power of \( \epsilon \). The equations corresponding to \( O(\epsilon) \), which include only the linear terms, are the same as those in the case of longitudinal waves and are given by equation (4.1, 1). Solving these equations, imposing the conditions \( D_3(k, \omega) \neq 0, D_2(k, \omega) \neq 0 \) and \( D_3(k, \omega) = 0 \), we find the following results:

\[
p_{12}^{\dagger} = v_{12}^{\dagger} = \varphi_{12}^{\dagger} = \psi_{12}^{\dagger} = 0, \quad v_{21}^{\dagger} = -\frac{\omega}{k}p_{21}^{\dagger} \tag{2.2}
\]

\[
\varphi_{21}^{\dagger} = -(1)^{\alpha\beta}\varepsilon p_{12}^{\dagger}, \quad \psi_{21}^{\dagger} = -(1)^{\alpha\beta}\varepsilon \omega p_{12}^{\dagger}, \quad \alpha \neq \beta \tag{2.3}
\]

As seen from these relations, all the first-order longitudinal wave amplitudes are zero whereas the remaining non-zero transverse wave amplitudes are known in terms of the amplitudes of two linearly polarized transverse waves, namely, \( p_{21}^{\dagger} \) and \( p_{31}^{\dagger} \), which will be determined from higher-order perturbation problems.

For \( O(\epsilon^2) \) we obtain

\[
\sum_{l=-\infty}^{\infty} \{ \rho_{0l}(kc_s^3p_{12}^{\dagger} + \omega v_{12}^{\dagger}) + \sum_{j=-\infty}^{\infty} [a_2(\varphi_{21}^{\dagger}\psi_{21}^{\dagger} + \varphi_{31}^{\dagger}\psi_{31}^{\dagger}) + a_3(\psi_{21}^{\dagger}p_{21}^{\dagger} + ik\varphi_{21}^{\dagger}p_{21}^{\dagger} - \varphi_{21}^{\dagger}p_{21}^{\dagger} - ik\varphi_{21}^{\dagger}p_{21}^{\dagger}) \\
+ ijk a_2(\psi_{21}^{\dagger}p_{21}^{\dagger} + p_{31}^{\dagger}p_{31}^{\dagger}) + ika_2(\psi_{21}^{\dagger}\psi_{31}^{\dagger} + \varphi_{21}^{\dagger}\varphi_{31}^{\dagger})] \epsilon^{\alpha\beta} \epsilon^{\alpha\beta} = 0
\]

\[
\sum_{l=-\infty}^{\infty} \left[ c_s^3 \left( \frac{\partial^2 p_{21}^{\dagger}}{\partial \xi^2} + ilkp_{21}^{\dagger} \right) - (1)^{\alpha\beta}\varepsilon \omega_0^2 \psi_{21}^{\dagger} + \Lambda \frac{\partial^2 \varphi_{21}^{\dagger}}{\partial \xi^2} + il\omega \psi_{21}^{\dagger} \right] \epsilon^{\alpha\beta} = 0, \quad \alpha \neq \beta
\]

\[
\sum_{l=-\infty}^{\infty} \left[ \rho_0(ilk c_s^3 p_{12}^{\dagger} - \omega_0^2 \psi_{12}^{\dagger} + il\omega \psi_{12}^{\dagger}) + \sum_{j=-\infty}^{\infty} \left[ \frac{1}{2} (a_{10} + a_{1} - a_{1}) (l + j) k a_2 (p_{21}^{\dagger} p_{31}^{\dagger} + p_{31}^{\dagger} p_{21}^{\dagger}) + (a_{10} + a_{1} - \frac{1}{2}) (\varphi_{21}^{\dagger} p_{21}^{\dagger} + p_{31}^{\dagger} p_{31}^{\dagger}) \\
+ \frac{\rho_0}{2} (il \omega (\varphi_{21}^{\dagger} w_{21}^{\dagger} - \varphi_{31}^{\dagger} w_{31}^{\dagger})) \right] \epsilon^{\alpha\beta} \epsilon^{\alpha\beta} = 0
\]

\[
\sum_{l=-\infty}^{\infty} \left[ b_s^3 \left( \frac{\partial^2 \varphi_{21}^{\dagger}}{\partial \xi^2} + ilk \varphi_{21}^{\dagger} - \omega_0^2 \varphi_{21}^{\dagger} + \frac{1}{2} \omega_0^2 \psi_{31}^{\dagger} + \Lambda \frac{\partial^2 \varphi_{21}^{\dagger}}{\partial \xi^2} + il\omega \psi_{31}^{\dagger} \right] \epsilon^{\alpha\beta} = 0, \quad \alpha \neq \beta
\]
\[ \sum_{j=-\infty}^{\infty} \left( \frac{\partial p_{k1}'}{\partial \xi} + i \omega p_{k2} + \frac{\partial v_{k1}'}{\partial \xi} + i k v_{k2}' \right) e^{i\omega_0} = 0 \]
\[ \sum_{j=-\infty}^{\infty} \left( \frac{\partial \psi_{k1}'}{\partial \xi} + i \omega \psi_{k2} + \frac{\partial w_{k1}'}{\partial \xi} + i k w_{k2}' \right) e^{i\omega} = 0 \]
\[ \sum_{j=-\infty}^{\infty} \left( \frac{\partial \varphi_{k1}'}{\partial \xi} + i \omega \varphi_{k2} + w_{k2}' \right) e^{i\omega} = 0 \]  \hspace{1cm} (2.4)

We solve these equations using the results in equations (2.2) and find

\[ l = 0: \quad \psi_{l2}^0 = 0, \quad \psi_{l2}^0 = 0, \quad w_{l2}^0 = 0, \quad p_{l2}^0 = (-1)^{\alpha} 2 \varphi_{l2}^0, \quad \alpha \neq \beta \]
\[ l = 1: \quad p_{l2}^1 = \psi_{l2}^1 = \psi_{l2}^1 = w_{l2}^1 = 0, \quad v_{l2}^1 = -\frac{\omega}{k} p_{l2}^1 + \frac{1}{k} \left( \Lambda - \frac{\omega}{k} \right) \frac{\partial p_{l1}^1}{\partial \xi} \]
\[ \alpha \neq \beta \]
\[ \varphi_{l2}^1 = -(-1)^{\alpha} \left[ c_p \frac{\partial p_{l2}^1}{\partial \xi} + \frac{2 i \omega}{c_0^2 k} \left( \Lambda - \frac{\omega}{k} \right) \frac{\partial p_{l1}^1}{\partial \xi} \right] \]
\[ \alpha \neq \beta \]
\[ \psi_{l2}^2 = -(-1)^{\alpha} \left[ i k \varphi_{l2}^1 + \left[ \frac{2 \omega}{c_0^2 k} \left( \Lambda - \frac{\omega}{k} \right) + c \right] \frac{\partial p_{l1}^1}{\partial \xi} \right] \]
\[ \alpha \neq \beta \]
\[ w_{l2}^1 = (-1)^{\alpha} \left[ i k \varphi_{l2}^1 + \left[ \frac{2 \omega}{c_0^2 k} \left( \Lambda - \frac{\omega}{k} \right) + c \right] \frac{\partial p_{l1}^1}{\partial \xi} \right] \]
\[ \alpha \neq \beta \]
\[ l = 2: \quad p_{l2}^2 = v_{l2}^2 = 0, \quad \varphi_{l2}^2 = 0, \quad \psi_{l2}^2 = w_{l2}^2 = 0 \]
\[ p_{l2}^1 = \frac{k^2 \left[ c_p^2 (a_2 - a_3 k^2) + 2 a_3 c + a_3 \right]}{2 \rho_0 (\omega^2 - c_0^2 k^2)} \left( \frac{p_{l1}^1}{\omega} + \frac{p_{l1}^1}{\omega} \right) \]
\[ \alpha \neq \beta \]
\[ \frac{2}{2 \omega} \left( \frac{p_{l1}^1}{\omega} + \frac{p_{l1}^1}{\omega} \right) \]
\[ \alpha \neq \beta \]
\[ p_{l2}^0 = v_{l2}^0 = \varphi_{l2}^0 = 0, \quad p_{l2}^0 = \frac{c_0^2 (a_2 + a_3 k^2) + 2 a_3 c + a_3}{2 \rho_0 (c_0^2 - c_0^2 k^2)} ((p_{l1}^1)^2 + (p_{l1}^1)^2) \]  \hspace{1cm} (2.5)

where all higher-order terms vanish for \( l \geq 3 \). In the solution of equations (2.4) for \( l = 1 \) we obtain the expressions \( (\Lambda - c_0) \partial p_{l1}^1 / \partial \xi = 0 \). Again, to obtain nontrivial solutions, we require that the relation \( \Lambda = c_0 \) must be satisfied.

We now consider the third-order problem. As the explicit forms of the equations at \( O(\epsilon^3) \) are too complicated and there will be no need to know their explicit forms, we will not present here all the third-order equations. For \( l = 0 \), we find that

\[ p_{l2}^0 = v_{l2}^0 = \varphi_{l2}^0 = 0, \quad p_{l2}^0 = \frac{c_0^2 (a_2 + a_3 k^2) + 2 a_3 c + a_3}{2 \rho_0 (c_0^2 - c_0^2 k^2)} ((p_{l1}^1)^2 + (p_{l1}^1)^2) \]  \hspace{1cm} (2.6)

It is interesting to note that although no longitudinal wave exists in the lowest-order problem [see the results of \( O(\epsilon) \)], nonlinear self-interaction of transverse waves in the lowest order can generate a longitudinal displacement wave in the second order of approximation [see the expressions of \( p_{l2}^0 \) in equations (2.6) and \( p_{l2}^1 \) in equations (2.5)]. For the first mode \( (l = 1) \) of \( O(\epsilon^3) \), considering the equations related to transverse components only, from equations (3.2.1), we obtain the following equations:

\[ \rho_0 \left[ b_1 \left( \frac{\partial p_{l2}^1}{\partial \xi} + k \varphi_{l2}^1 \right) - (-1)^{\alpha} c_0 \varphi_{l2}^1 + c_0 \frac{\partial v_{l2}^1}{\partial \xi} - \frac{\partial w_{l2}^1}{\partial \xi} + i \omega v_{l2}^1 \right] 
\[ + (-1)^{\alpha} a_3 (p_{l2}^1 p_{l2}^1 + \varphi_{l2}^1 p_{l2}^1 + 2 i k \varphi_{l2}^1 p_{l2}^1) + 2 i k \varphi_{l2}^1 p_{l2}^1 + i k \omega p_{l2}^1 p_{l2}^1 + i \omega p_{l2}^1 p_{l2}^1 = 0, \]
\[ \alpha \neq \beta \]
\[ \rho_0 k \left[ b_1 \left( \frac{\partial \psi_{l2}^1}{\partial \xi} + k \psi_{l2}^1 \right) - \omega_0 \varphi_{l2}^1 - \frac{(-1)^{\alpha}}{2} \omega_0 \varphi_{l2}^1 + c_0 \frac{\partial w_{l2}^1}{\partial \xi} + \frac{\partial w_{l2}^1}{\partial \xi} + i \omega w_{l2}^1 \right] 
\[ - a_2 (p_{l2}^1 \varphi_{l2}^1 + p_{l2}^0 \varphi_{l2}^1) + (-1)^{\alpha} a_3 (p_{l2}^1 p_{l2}^1 + p_{l2}^1 p_{l2}^1) + i k \omega (p_{l2}^1 p_{l2}^1 + p_{l2}^1 p_{l2}^1) = 0, \]
\[ \alpha \neq \beta \]
\[ v_{l3} = -\frac{\omega}{k} p_{l3} + \frac{i}{k} \left( c_0 - \frac{\omega}{k} \right) \frac{\partial \psi_{l2}^1}{\partial \xi} - \frac{i}{k} \left( c_0 - \frac{\omega}{k} \right) \frac{\partial \psi_{l2}^1}{\partial \xi} \]
\[ \psi_{l3} = i k \varphi_{l3} + \frac{\partial \varphi_{l2}^1}{\partial \xi}, \quad w_{l3} = -i k \varphi_{l2}^1 - c_0 \frac{\partial \varphi_{l1}^1}{\partial \xi} + \frac{\partial \varphi_{l1}^1}{\partial \xi} \]  \hspace{1cm} (2.7)
If the higher-order terms are removed from the first two of equations (2.7), using the remaining equations [equations (2.7)\_3], after some lengthy calculations, the following two coupled equations are obtained for the amplitudes of transverse displacement waves, \( p_{21} \) and \( p_{31} \):

\[
\begin{align*}
\frac{\partial^2 \Phi}{\partial \tau^2} + \Gamma_T \frac{\partial^2 \Phi}{\partial \xi^2} + \Delta_1 \Phi (|\Phi|^2 + |\Psi|^2) + \Delta_2 \Phi^* (|\Phi|^2 + |\Psi|^2) &= 0 \\
\frac{\partial^2 \Psi}{\partial \tau^2} + \Gamma_T \frac{\partial^2 \Psi}{\partial \xi^2} + \Delta_1 \Psi (|\Phi|^2 + |\Psi|^2) + \Delta_2 \Psi^* (|\Phi|^2 + |\Psi|^2) &= 0 \tag{2.8}
\end{align*}
\]

where \( \Phi = p_{21} \) and \( \Psi = p_{31} \), and

\[
\Gamma_T = \frac{d^2 \omega}{d \kappa^2} = \frac{\rho_0}{\Delta_0} \left( \frac{9k^2}{\omega_0^2} \left[ -2b_2 c_s^2 + \frac{2b_2^2 \omega}{k} \left( 4c_s^2 - \frac{3\omega}{k} \right) + \frac{2c_s^4 - \omega^2}{k^2} \right] - \frac{8c_s^2 \omega^2}{k^2} \left( c_5 - \frac{\omega}{k} \right) \right)
\]

\[
\Delta_1 = \frac{k^2}{\rho_0 (c_s^2 - c_L^2)} [ \omega^2 (a_2 + a_3 \kappa^2) + 2a_3 \xi + a_5 ]^2
\]

\[
\Delta_2 = \frac{k^4}{2\rho_0 (\omega^2 - c_L^2 \kappa^2)} [ \omega^2 (a_2 - a_3 \kappa^2) + 2a_3 \xi + a_5 ]^2 \tag{2.9}
\]

where

\[
\Delta_0 = -\frac{2\rho_0 \omega}{\omega_0 c_L^2 k^2} [ \omega^2 c_L^2 \kappa^2 + (c_L^2 \kappa^2 - \omega^2)^2 ] \tag{2.10}
\]

Equations (2.8) describe the variation of the complex amplitudes of the interacting waves with linear polarization along the \( X_2 \) and \( X_3 \)-directions. It should be noted that the coefficient \( \Delta_1 \) or \( \Delta_2 \) in equations (2.9) becomes infinitely large when \( \omega^2 = c_L^2 \kappa^2 \) or \( c_s = c_L \), respectively. The interaction equations [equations (2.8)] then break down. Noting that we have already assumed \( \omega^2 \neq c_L^2 \kappa^2 \) to avoid such a case we further require that \( c_s \neq c_L \). These evolution equations [equations (2.8)], describing the nonlinear interaction of the two linearly polarized transverse waves, are also valid, with slightly different coefficients, for the amplitudes of the transverse microrotation waves, \( \varphi_{21} \) and \( \varphi_{31} \), because of the relations in equations (2.2).

Recalling equations (3.1,1) and (3.5,1), all the first-order transverse components can be written in terms of the complex amplitudes \( \Phi \) and \( \Psi \) as follows:

\[
\begin{align*}
\varphi_{21} &= \Phi e^{i \theta} + \Phi^* e^{-i \theta}, & \quad p_{21} &= \Psi e^{i \theta} + \Psi^* e^{-i \theta} \\
\varphi_{31} &= -\tilde{\xi} (\Psi e^{i \theta} + \Psi^* e^{-i \theta}), & \quad \varphi_{31} &= \Phi e^{i \theta} + \Phi^* e^{-i \theta} \tag{2.11}
\end{align*}
\]

for which the displacement gradient vector takes the form \( \partial \mathbf{u} / \partial X = \epsilon (p_{21} I_2 + p_{31} I_3) + O(\epsilon^2) \), where \( I_2 \) and \( I_3 \) are unit vectors in the \( X_2 \) and \( X_3 \)-directions, respectively. It should be noted that although the longitudinal displacement gradient is zero in the first order of the approximation, in general, it is different from zero in the second order as a result of nonlinear self-interaction of transverse waves. To examine the properties of the solutions it is more convenient to rewrite equations (2.8) in terms of the complex amplitudes of two circularly polarized transverse waves. For this aim we make the following transformation:

\[
\begin{align*}
p_- &= (p_{21} + ip_{31}) / \sqrt{2}, & \quad p_+ &= (p_{21} - ip_{31}) / \sqrt{2} \tag{2.12}
\end{align*}
\]

The displacement gradient vector in the first-order approximation then takes the form

\[
\frac{\partial \mathbf{u}}{\partial X} = \epsilon \left[ p_- \left( \frac{I_2 - iI_3}{\sqrt{2}} \right) + p_+ \left( \frac{I_2 + iI_3}{\sqrt{2}} \right) \right] \tag{2.13}
\]

and \( p_- \) and \( p_+ \) can be written as

\[
\begin{align*}
p_- &= \psi e^{i \theta} + \psi^* e^{-i \theta}, & \quad p_+ &= \phi e^{i \theta} + \phi^* e^{-i \theta} \tag{2.14}
\end{align*}
\]
where $\phi$ and $\psi$ are the complex amplitudes of the two circularly polarized waves and are defined by
\[
\psi = (\Phi + i\Psi)/\sqrt{2}, \quad \phi = (\Phi - i\Psi)/\sqrt{2}.
\] (2.15)

Thus, by making use of this transformation in equations (2.8), we reach the following two coupled NLS equations:
\[
i \frac{\partial \phi}{\partial \tau} + \Gamma_1 \frac{\partial^2 \phi}{\partial \xi^2} + [\Delta_1 |\phi|^2 + (\Delta_1 + 2\Delta_2) |\psi|^2] \phi = 0
\]
\[
i \frac{\partial \psi}{\partial \tau} + \Gamma_1 \frac{\partial^2 \psi}{\partial \xi^2} + [\Delta_1 |\psi|^2 + (\Delta_1 + 2\Delta_2) |\phi|^2] \psi = 0
\] (2.16)

where the second terms represent the dispersive effect, whereas the first and second terms in brackets indicate the self- and mutual interactions, respectively. If one of the two waves does not exist or their amplitudes are equal, this system reduces to the well-known single NLS equation. Moreover, it can easily be shown that each wave conserves its energy:
\[
\frac{d}{d\tau} \int |\phi|^2 \, d\xi = \frac{d}{d\tau} \int |\psi|^2 \, d\xi = 0
\]
where we have assumed that $\phi$ and $\psi$ decrease sufficiently rapidly for arbitrarily large $|\xi|$. This conclusion implies that, if a particular wave does not exist initially, it does not exist at all, i.e., one wave cannot generate the other. On the other hand, for linearly polarized waves corresponding to equations (2.8), the energy of each wave is not usually conserved, and then a wave can generate another.

These coupled NLS equations are equivalent to those obtained by Berkhoeer and Zakharov [3] in their study of nonlinear electromagnetic waves in an isotropic plasma. Similar coupled evolution equations have also been obtained for a wide class of dispersive wave systems in many branches of physics and engineering (for instance, by Inoue [4, 5], Spatschek [6], Som et al. [7], Yoshinaga et al. [8], Menyuk [9], Newboult et al. [10] and Teymur [11]). The complete integrability of these equations, for the specific parametric restrictions, has been established by Zakharov and Schulman [12] in terms of “motion invariants”. Later similar results have been obtained by Sahadevan et al. [13] via Painlevé analysis. According to these results, in our case, the condition $\Delta_2 = 0$ corresponds to the integrability of these coupled NLS equations. Using this condition, which is equivalent to setting the terms in the bracket in the expression of $\Delta_2$ in equations (2.9) equal to zero, and recalling the definition of $\bar{c}$ in equation (2.3), we obtain
\[
\omega^2(a_2 - a_1k^2) - 2\omega^2k^2[c^2(\alpha_2 - \alpha_1k^2) + c_0^2a_3] + k'[c^2(\alpha_2 - \alpha_1k^2) + 2a_2c^2c_0^2 + a_3c_0^2] = 0
\] (2.18)

As the pair $(k, \omega)$ also has to satisfy the dispersion relation $D_3(k, \omega) = 0$, then the integrability condition is fulfilled only at the mutual points of the dispersion curves corresponding to $D_3(k, \omega) = 0$ and the curves corresponding to equation (2.18) in the plane $(k, \omega)$. Further, if we restrict ourselves to the geometric nonlinearity only, i.e. $a_7 = 0$ because of equations (2.17, I), as the terms in brackets in the expressions of $\Delta_1$ and $\Delta_2$ in equations (2.9) will be the same, the integrability condition, $\Delta_2 = 0$, will also require $\Delta_1 = 0$. This implies that the integrability condition will be satisfied only when these nonlinear interaction equations reduce to two linear uncoupled equations.

2.1 Some special solutions

In [3], Berkhoeer and Zakharov sought the solutions of these coupled NLS equations in the form
\[
\phi = U(\xi) \exp[i\Theta_1(\xi, \tau)], \quad \psi = V(\xi) \exp[i\Theta_2(\xi, \tau)]
\]
(2.19)

where $U$, $V$, $\Theta_1$ and $\Theta_2$ are real functions. Inserting these solutions into equations (2.16), we obtain four coupled equations for $U$, $V$, $\Theta_1$ and $\Theta_2$. It can easily be verified that $\Theta_1$ and $\Theta_2$ must have the forms $\Theta_1 = f_1(\xi) - \Omega_1 \tau$ and $\Theta_2 = f_2(\xi) - \Omega_2 \tau$, where $\Omega_1$ and $\Omega_2$ are arbitrary
constants, and the following differential equations for $\Theta_1$ and $\Theta_2$ must be satisfied:

$$U^2 \frac{d\Theta_1}{d\xi} = C_1, \quad V^2 \frac{d\Theta_2}{d\xi} = C_2$$

(2.20)

Using these results in the remaining equations, the following simultaneous ordinary differential equations for $U$ and $V$ are obtained:

$$\frac{d^2 U}{d\xi^2} = -\frac{V}{U} \frac{dV}{d\xi}, \quad \frac{d^2 V}{d\xi^2} = -\frac{U}{V} \frac{dU}{d\xi}$$

(2.21)

where

$$\gamma(U, V) = \frac{1}{2\Gamma_T} \left[ \Omega_1 U^2 + \Omega_2 V^2 + \frac{\Delta_1}{2} (U^4 + V^4) + (\Delta_1 + 2\Delta_2) U^2 V^2 + \frac{\Omega_1^2}{U^2} + \frac{\Omega_2^2}{V^2} \right]$$

(2.22)

These equations are formally analogous to the equations of two-dimensional motion of a particle in a field with potential $\gamma(U, V)$ and $\xi$ plays the role of time. In this dynamical system the energy is locally conserved, that is,

$$\frac{1}{2} \left[ \left( \frac{dU}{d\xi} \right)^2 + \left( \frac{dV}{d\xi} \right)^2 \right] + \gamma(U, V) = E(\text{const.})$$

(2.23)

The solutions to equations (2.21) can exist only within the region $\gamma(U, V) \leq E$, so that all the solutions are found to be bounded. In [4], Inoue, solving equations (2.21) numerically, showed that the coupled NLS equations have, in general, dispersive shock wave and non-periodic nonlinear wavetrains solutions which have no counterpart in the solutions of the single NLS equation as the travelling wave solutions.

We now consider the following simple classes of the travelling wave solutions of the problem.

2.1.1 Uniform wavetrains

It is easily seen that the following plane waves:

$$\phi(\xi, \tau) = U_0 \exp[i(K_1\xi - \Omega_1\tau)], \quad \psi(\xi, \tau) = V_0 \exp[i(K_2\xi - \Omega_2\tau)]$$

(2.24)

satisfy equations (2.16), provided that

$$\Omega_1 = \Gamma_T K_1^2 - \Delta_1 U_0^2 - (\Delta_1 + 2\Delta_2) V_0^2, \quad \Omega_2 = \Gamma_T K_2^2 - \Delta_1 V_0^2 - (\Delta_1 + 2\Delta_2) U_0^2$$

(2.25)

where $U_0$, $V_0$, $K_1$ and $K_2$ are real constants. These solutions represent two wavetrains with wave speeds dependent on the constant amplitudes $U_0$ and $V_0$. If the solutions (2.24) are substituted into equations (2.11) using (2.15), the displacement gradient components $p_2$ and $p_3$ are found in the first-order approximation in terms of the original variables as

$$p_2 = \sqrt{2}\varepsilon [U_0 \cos(\bar{\kappa}_1 X - \bar{\omega}_1 t) + V_0 \cos(\bar{\kappa}_2 X - \bar{\omega}_2 t)]$$

$$p_3 = -\sqrt{2}\varepsilon [U_0 \sin(\bar{\kappa}_1 X - \bar{\omega}_1 t) - V_0 \sin(\bar{\kappa}_2 X - \bar{\omega}_2 t)]$$

(2.26)

where

$$\bar{\kappa}_1 = k + \varepsilon K_1, \quad \bar{\omega}_1 = \omega + \varepsilon c_k K_1 + \varepsilon^2 \Omega_1$$

$$\bar{\kappa}_2 = k + \varepsilon K_2, \quad \bar{\omega}_2 = \omega + \varepsilon c_k K_2 + \varepsilon^2 \Omega_2$$

(2.27)

These expressions together with equations (2.25) show how nonlinearity gives rise to interaction between two circularly polarized waves with wave speeds $\bar{\omega}_1/\bar{\kappa}_1$ and $\bar{\omega}_2/\bar{\kappa}_2$. It should be noted that when the coupling coefficient $\Delta_1 + 2\Delta_2$ is zero these two circularly polarized waves do not interact each other.

When $K_1 = K_2$, for which $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\kappa}$, the solutions given in equations (2.26) take the following forms:

$$p_2 = 2\sqrt{2}\varepsilon U_0 \cos(\bar{\kappa} X - \bar{\omega} t), \quad p_3 = 0$$

(2.28)
when \( U_0 = V_0 \), and
\[
p_2 = 0, \quad p_3 = 2\sqrt{2}\varepsilon V_0 \sin(\hat{k}X - \hat{\omega}t)
\] (2.29)
when \( U_0 = -V_0 \). For both cases \( \hat{\omega}_1 = \hat{\omega}_2 = \hat{\omega} \). Equations (2.28) and (2.29) represent transverse nonlinear waves with linear polarization along the \( X_T^f \) and \( X_T^t \) directions, respectively.

Moreover, when \( K_1 = K_2 \) and \( \Delta_2 = 0 \), \( \hat{k}_1 = \hat{k}_2 = \hat{k} \) and \( \hat{\omega}_1 = \hat{\omega}_2 = \hat{\omega} \), and equations (2.26) become
\[
p_2 = \sqrt{2}\varepsilon(U_0 + V_0)\cos(\hat{k}X - \hat{\omega}t), \quad p_3 = -\sqrt{2}\varepsilon(U_0 - V_0)\sin(\hat{k}X - \hat{\omega}t)
\] (2.30)
In this case, the tip of the displacement gradient vector will be located on an ellipse.

2.1.2 Envelope solitary waves

2.1.2.1 Circularly polarized waves. If \( \phi \) is taken as zero, equations (2.16) reduce to the single NLS equation
\[
i\frac{\partial \psi}{\partial \tau} + \Gamma_T \frac{\partial^2 \psi}{\partial \xi^2} + \Delta_1 |\psi|^2 \psi = 0
\] (2.31)
whose solutions are given in [1]. Under the assumption of \( \Gamma_T \Delta_1 > 0 \), the envelope solitary wave solution of this equation is
\[
\psi = A \sech \left( \frac{\Delta_1}{2\Gamma_T} \right)^{1/2} A_\xi e^{i(k^2 - \Omega \tau)}
\] (2.32)
where \( \xi = \xi - 2\Gamma_T K \tau \) and \( \Omega = \Gamma_T K^2 - \Delta_1 A^2/2 \), and \( K \) and \( A \) are real constants. In this case, the displacement gradient components are given in terms of the original variables
\[
p_2 = \sqrt{2}\varepsilon A \sech \left( \frac{\Delta_1}{2\Gamma_T} \right)^{1/2} A(X - v_\xi t) \cos(\hat{k}X - \hat{\omega}t)
\]
\[
p_3 = \sqrt{2}\varepsilon A \sech \left( \frac{\Delta_1}{2\Gamma_T} \right)^{1/2} A(X - v_\xi t) \sin(\hat{k}X - \hat{\omega}t)
\] (2.33)
where
\[
\hat{k} = k + \varepsilon K, \quad \hat{\omega} = \omega + \varepsilon c_\xi K + \varepsilon^2 \Omega, \quad v_\xi = c_\xi + 2\varepsilon \Gamma_T K
\] (2.34)
Equations (2.33) represent a circularly polarized transverse wave whose envelope is a solitary wave.

On the other hand, it can easily be seen that when \( \psi = 0 \) the corresponding solutions can be obtained by substituting \((p_2, -p_3)\) for \((p_2, p_3)\) in equations (2.33).

2.1.2.2 Linearly polarized waves. When we take \( \phi = \psi \) or \( \phi = -\psi \), in both cases equations (2.16) reduce to the single NLS equation
\[
i\frac{\partial \psi}{\partial \tau} + \Gamma_T \frac{\partial^2 \psi}{\partial \xi^2} + 2(\Delta_1 + \Delta_2) |\psi|^2 \psi = 0
\] (2.35)
which includes the coefficient \( \Delta_1 + 2\Delta_2 \) describing the mutual interaction between the two waves. Similarly, the envelope solitary wave solution of this equation is written for \( \Gamma_T(\Delta_1 + \Delta_2) > 0 \) as follows:
\[
\psi = A \sech \left( \frac{\Delta_1 + \Delta_2}{\Gamma_T} \right)^{1/2} A_\xi e^{i(k^2 - \Omega \tau)}
\] (2.36)
where \( \Omega = \Gamma_T K^2 - (\Delta_1 + \Delta_2) A^2 \), and \( \xi, K \) and \( A \) are defined as in the preceding case. The corresponding displacement gradient components are calculated in terms of the original variables as, for \( \phi = \psi \),
\[
p_2 = 2\sqrt{2}\varepsilon A \sech \left( \frac{\Delta_1 + \Delta_2}{\Gamma_T} \right)^{1/2} A(X - v_\xi t) \cos(\hat{k}X - \hat{\omega}t), \quad p_3 = 0
\] (2.37)
and, for $\phi = -\psi,$

$$p_2 = 0, \quad p_3 = 2\sqrt{2}eA \text{sech}\left[\left(\frac{\Delta_1 + \Delta_2}{\Gamma_T}\right)^{1/2} \epsilon A(X - v_xt)\right] \sin(kX - \bar{\omega}_2 t) \tag{2.38}$$

where $\bar{k}, \bar{\omega}$ and $v_x$ are given in equations (2.34). Equations (2.37) and (2.38) describe transverse waves with "sech envelopes", polarized linearly in the $X_T$ and $X_3$-directions, respectively.

Recalling the discussion in [1] on the solutions of the NLS equation, we find that "sech envelopes" in the above solutions become "tanh envelopes" for circularly polarized waves when $\Gamma_T \Delta_1 < 0,$ and for linearly polarized waves when $\Gamma_T (\Delta_1 + \Delta_2) < 0.$

As a final remark of this subsection, we should point out that the coupled NLS equations also have a set of solutions consisting of a solitary wave and a phase jump given by

$$\phi = A \text{sech}\left[\left(\frac{\Delta_1}{2\Gamma_T}\right)^{1/2} (A^2 - B^2)^{1/2} \epsilon \right] \exp[i(K\xi - \Omega_1 \tau)],$$

$$\psi = B \tanh\left[\left(\frac{\Delta_1}{2\Gamma_T}\right)^{1/2} (A^2 - B^2)^{1/2} \epsilon \right] \exp[i(K\xi - \Omega_2 \tau)] \tag{2.39}$$

which was obtained in [5] as a special case of the solutions expressed by Jacobian elliptic functions. Here $\Delta_1 = 0$ and $\xi = \xi - 2\Gamma_T K x,$ and

$$\Omega_1 = \Gamma_T K^2 - \Delta_1 (A^2 + B^2)/2, \quad \Omega_2 = \Gamma_T K^2 - \Delta_1 B^2 \tag{2.40}$$

where $K, A$ and $B$ are real constants such that $(A^2 - B^2) > 0.$ In this case, $p_2$ and $p_3$ are

$$p_2 = \sqrt{2}e\left\{A \text{sech}\left[\left(\frac{\Delta_1}{2\Gamma_T}\right)^{1/2} (A^2 - B^2)^{1/2} \epsilon (X - v_xt)\right] \cos(\bar{k}X - \bar{\omega}_1 t)\right\}$$

$$+ B \tanh\left[\left(\frac{\Delta_1}{2\Gamma_T}\right)^{1/2} (A^2 - B^2)^{1/2} \epsilon (X - v_xt)\right] \cos(\bar{k}X - \bar{\omega}_2 t)\right\}$$

$$p_3 = -\sqrt{2}e\left\{A \text{sech}\left[\left(\frac{\Delta_1}{2\Gamma_T}\right)^{1/2} (A^2 - B^2)^{1/2} \epsilon (X - v_xt)\right] \sin(\bar{k}X - \bar{\omega}_1 t)\right\}$$

$$- B \tanh\left[\left(\frac{\Delta_1}{2\Gamma_T}\right)^{1/2} (A^2 - B^2)^{1/2} \epsilon (X - v_xt)\right] \sin(\bar{k}X - \bar{\omega}_2 t)\right\} \tag{2.41}$$

where $\bar{k}, \bar{\omega}_1$ and $\bar{\omega}_2$ are defined in equations (2.27) with $\bar{k}_1 = \bar{k}_2 = \bar{k}.$ These solutions represent how nonlinearity gives rise to interaction between two circularly polarized transverse waves with wave speeds $\bar{\omega}_1/\bar{k}$ and $\bar{\omega}_2/\bar{k}$ when one of these waves has a "sech envelope" and the other has a "tanh envelope". It should be noted that by making use of the symmetry between $\phi$ and $\psi,$ one can obtain a similar type of solution.

2.2 Modulational instability

To study the modulational instability of the coupled waves with constant amplitude, we now consider the stability of the plane wave solutions (2.24) under the infinitesimal modulational perturbation. As mentioned in [1], a constant-amplitude plane wave solution of the single NLS equation is modulationally stable if the coefficients of dispersive and nonlinear terms take opposite signs. However, in the case of two coupled waves propagating in the medium, it is not easy, in general, to determine whether the waves are modulationally stable or not. We therefore consider the stability of such coupled waves under some special conditions. To this end, following the standard procedure, we consider the modulational perturbations of the coupled waves in the form

$$\phi = (U_0 + \bar{U}) \exp[i(K_1 \xi - \Omega_1 \tau) + i\bar{\Theta}_1], \quad \psi = (V_0 + \bar{V}) \exp[i(K_2 \xi - \Omega_2 \tau) + i\bar{\Theta}_2] \tag{2.42}$$

where $\bar{U}, \bar{V}, \bar{\Theta}_1$ and $\bar{\Theta}_2$ are small perturbations. $\Omega_1$ and $\Omega_2$ appearing in equations (2.42) must satisfy the relations given in equations (2.25). Inserting these solutions [equations (2.42)] into the coupled NLS equations [equations (2.16)] and neglecting nonlinear terms with respect to
the perturbations and seeking harmonic wave solutions of the linearized equations, we reach
the following dispersion relation for long waves:

\[
[(\Omega - 2\Gamma_J K_1 K)^2 + 2\Delta_1 \Gamma_J U_0^2 K^2][(\Omega - 2\Gamma_J K_2 K)^2 + 2\Delta_1 \Gamma_J V_0^2 K^2] - 4\Gamma_J^2(\Delta_1 + 2\Delta_2)^2 U_0^2 V_0^2 K^4 = 0
\]

(2.43)

where \( \Omega \) and \( K \) denote the frequency- and wavenumber of the perturbations, respectively. When \( \Delta_1 + 2\Delta_2 \) is sufficiently small, the last term of equation (2.43) can be neglected. In this case, the coupling disappears and equation (2.43) reduces to the well-known dispersion relations for the uncoupled waves. Thus the plane wave solutions are modulationally unstable when \( \Gamma_J \Delta_1 > 0 \) and stable when \( \Gamma_J \Delta_1 < 0 \). For a special case where \( K_1 = K_2 \), the above dispersion relation reduces to a quadratic equation whose roots are

\[
(\Omega - 2\Gamma_J K_1 K)^2 = K^2 \Gamma_J (-\Delta_1(U_0^2 + V_0^2) + [\Delta_1(U_0^2 + V_0^2)^2 + 16\Delta_2(\Delta_1 + \Delta_2)U_0^2 V_0^2]^{1/2})
\]

(2.44)

Let us first assume that \( \Delta_2(\Delta_1 + \Delta_2) > 0 \). Then the root corresponding to the lower (or upper)
sign is negative when \( \Gamma_J < 0 \) (or \( \Gamma_J > 0 \)), and thus the instability occurs. Similarly, when \( \Delta_2(\Delta_1 + \Delta_2) < 0 \) and the inside of the square root is positive, it can readily be seen that the coupled waves are stable if \( \Gamma_J \Delta_1 < 0 \) or unstable if \( \Gamma_J \Delta_1 > 0 \). Moreover, for the limiting case \( U_0 = V_0 \) together with \( K_1 = K_2 \), the coupled waves are stable if \( \Gamma_J \Delta_2 > 0 \) and \( \Gamma_J (\Delta_1 + \Delta_2) < 0 \).

The above observations about the modulational instability show that the condition stabilizing
the wave system becomes more restricted than that for the case where only one wave exists,
and the nonlinear coupling tends to give rise to the instability of the two waves.

3. CONCLUSIONS

In this paper, we have considered the amplitude modulation of transverse plane waves
propagating in an infinite micropolar elastic medium and showed that the modulation can be
described by two coupled NLS equations in terms of the amplitudes of the two circularly
polarized transverse displacement waves. A similar system of coupled evolution equations is also
valid for the amplitudes of the transverse microrotation waves. The crucial point of the
present analysis is that the linearized transverse waves have two independent amplitudes and
that both waves have an identical dispersion relation. We therefore point out that this
approach, leading to two coupled nonlinear interaction equations, should not be confused with
the harmonic resonance cases which result from the resonance between the fundamental and its
higher harmonic modes whose phase velocities coincide with each other at a critical
wavenumber. For instance, if we consider transverse plane waves satisfying the resonance
conditions—say, the second harmonic resonance—we would expect to obtain four coupled
nonlinear interaction equations in which the energy conservation law may not hold for each
wave and energy can be transferred from wave to wave, whereas the total energy is conserved.
Finally, we also point out that it is not possible to obtain nonlinear evolution equations
considering pure transverse motions, i.e. when \( u_1 = \varphi_1 = 0 \). In this case, the coupled NLS
equations reduce to the linear uncoupled equations as all the nonlinear terms representing the self- and mutual interactions in the coupled NLS equations arise through the second-order
longitudinal displacement gradients \( p_{\perp 1}^0 \) and \( p_{\perp 2}^1 \). The reason for this situation is that the nonlinear terms in the expressions of the transverse stress and couple stress components drop in the case of pure transverse motions.

REFERENCES


(Received 27 September 1990; accepted 29 October 1990)