NONLINEAR INTERACTION OF TRANSVERSE ACOUSTICAL AND OPTICAL WAVES IN MICROPOLAR ELASTIC MEDIA

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Abstract—In this study nonlinear interaction between modulated transverse waves in a weakly nonlinear micropolar elastic medium is considered. The interaction is assumed to be between transverse acoustic and optical plane waves with equal group velocity. By using the reductive perturbation method, it is shown that the slow modulation of the complex envelopes is described by four coupled nonlinear evolution equations. As a special case, these equations reduce to a system of four coupled nonlinear Schrödinger equations. Some special solutions of the evolution equations, namely, nonlinear plane wave and envelope solitary wave solutions, are also presented.

1. INTRODUCTION

Various theories have been proposed in recent years to incorporate the internal, discrete, structure of matter into the classical elasticity model. These theories take different names depending on which aspect of the continuum has been chosen as a starting point. Higher-order gradients, nonlocal particle interactions, polyatomic structure and local intrinsic rotations are some of these aspects, among others. From the viewpoint of the wave phenomena, it is interesting that all these theories, contrary to the classical elasticity theory, allow us to observe dispersive wave propagation in a linear approximation. This feature of the models, i.e. their dispersive character, may give rise to coherent structures such as solitary waves if nonlinearity is included. When both dispersion and nonlinearity are present in the medium, the competition between the steepening as a result of nonlinearity and the spreading as a result of dispersion favors a travelling wave of constant profile and velocity. Such nonlinear stable excitations play an important role in physics and engineering and are found in various fields such as nonlinear optics, water waves, plasma physics, etc.

In this study one of the above-mentioned generalized continuum theories, i.e. micropolar elasticity theory which takes into account the local rigid microrotations of material particles, will be considered. The theory of micropolar elasticity is concerned with an elastic medium whose constituents, the so-called material points, are allowed to rotate independently without stretch. Hence, the motion of the material points of such a medium will have three additional degrees of freedom associated with local micro-rigid rotations. The fundamental equations of a micropolar elastic medium then contain coupled microrotation and displacement fields. The literature in this area is very extensive, and for details see the review articles by Eringen [1] (with special reference to the linear theories of micropolar elasticity) and by Eringen and Kafadar [2].

There are already some works on nonlinear wave propagation in micropolar elastic media (for instance, see the works of Maugin and Miled [3], Pouget and Maugin [4], Erbay and Suhubi [5] and Erbay et al. [6–8]). This study is a sequel to the earlier studies [6–8] about the amplitude modulation of longitudinal and transverse plane waves propagating in a weakly nonlinear micropolar elastic medium. In [6], using the reductive perturbation method, it was shown that the slowly varying complex amplitude of a longitudinal microrotation wave is described by the nonlinear Schrödinger (NLS) equation. As is known, the linear dispersion relation corresponding to transverse waves in micropolar elastic solids

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has two branches, the so-called acoustical branch which involves low frequencies and the optical branch which involves a range of higher frequencies. In [7], it was shown that, making no distinction between transverse acoustical and optical branches, the slowly varying complex amplitudes of the transverse (displacement and microrotation) waves are governed by two coupled NLS equations. The crucial point of the analysis in [7] is that the linearized transverse waves have two independent amplitudes and both waves have the same dispersion relation. In [8], the modulational instability of the plane wave solutions of the coupled NLS equations was discussed. In the present study, the nonlinear interaction between modulated transverse acoustical and optical plane waves in micropolar elastic media is investigated.

In recent years, much attention has been paid to nonlinear wave–wave interactions in dispersive media, which play an important role in exchanging energy among different wave modes. Such interactions occur strongly if some resonance conditions are satisfied with respect to frequencies and wavenumbers of different wave modes. The theory of wave–wave interactions provides the basic equations governing the interaction of several quasi-monochromatic waves, which describe the energy transfer between several modes belonging to resonant groups. Already there exist some works taking into account wave–wave interactions in a dispersive medium having two branches (see Dodd et al. [9]). Many of them consider the case where the group velocity of the optical wave coincides with the phase velocity of the acoustical wave, i.e. long wave–short wave interaction. However, the case where the group velocities of acoustical and optical waves are equal to each other, which is the subject of the present work, was considered only by Inoue [10], while he was working on weakly nonlinear electromagnetic waves in an isotropic dielectric and by Som et al. [11] while they were studying Langmuir and ion acoustic waves in the plasma. In these works, it was shown that the slow modulation of the complex amplitudes for two waves having equal group velocity is governed by two coupled NLS equations. On the other hand, in [7], it has been shown that, considering the transverse waves belonging to one of the branches only, the envelopes of transverse waves in micropolar elastic media satisfy similar equations due to the coupling between transverse components. Therefore, it will be interesting to consider the nonlinear interaction between transverse acoustical and optical waves having equal group velocity in micropolar elastic media. In such a case, for two different wavenumbers corresponding to acoustical and optical branches, there exist waves having equal group velocity in both branches.

In Section 2, the one-dimensional field equations derived in [6] for a weakly nonlinear micropolar elastic medium are briefly summarized. In the same section, the linear dispersion relations of plane waves, which reveal that the longitudinal microrotation and all transverse waves are dispersive, are presented. In Section 3, using the reductive perturbation method, it is shown that the slow modulation of the complex amplitudes of linearly polarized transverse (displacement and microrotation) waves is described by four coupled nonlinear evolution equations. Then, rewriting the system of these coupled nonlinear equations in terms of the amplitudes of circularly polarized transverse waves, it is shown that these equations reduce to a system of four coupled NLS equations with extra terms which represent energy exchange between the waves. In Section 4, the solutions corresponding to some special cases are presented, and the corresponding displacement gradient components are given in terms of the original variables. These solutions include circularly and linearly polarized waves with "sech envelope" and nonlinear wave trains. Section 5 is devoted to concluding remarks.

2. BASIC EQUATIONS AND DISPERSION RELATIONS

2.1. Basic equations

A micropolar elastic solid is affected by the local micro-rigid rotations of the material points. As every particle of a polar medium is assumed to behave like a rigid body, each point of such a medium can translate and rotate independently: Thus, a material point of such a medium has six degrees of freedom, three for translation and three for rotation. The translation and the rotation are described, respectively, as follows:

\[ x_k = x_k(X_K, t), \quad \varphi_k = \varphi_k(X_K, t) \quad k, K = 1, 2, 3, \]  
(2.1)
where $\varphi_k$ is the microrotation vector, and $x_k$ and $X_k$ are the spatial and material Cartesian coordinates of the same material point at times $t$ and $t = 0$, respectively.\(^1\)

For the study of one-dimensional plane waves, the displacement vector $u_k$ and the microrotation vector $\varphi_k$ are assumed to be functions of $X$ and $t$ only, i.e.

$$u_k = u_k(X, t), \quad \varphi_k = \varphi_k(X, t), \tag{2.2}$$

where $X \equiv X_1$ denotes the coordinate along the direction of propagation. For convenience we prefer to express the nonlinear field equations of micropolar elastic media as a system of first-order differential equations. Therefore, we introduce the derivatives of $u_k$ and $\varphi_k$ as new field variables, i.e.

$$p_k = \frac{\partial u_k}{\partial X}, \quad v_k = \frac{\partial u_k}{\partial t}, \quad \psi_k = \frac{\partial \varphi_k}{\partial X}, \quad w_k = \frac{\partial \varphi_k}{\partial t}. \tag{2.3}$$

The material forms of the one-dimensional balance equations with vanishing body forces and body couples are given for longitudinal and transverse directions by (for details, see [6]):

**Balance of linear momentum:**

$$\frac{\partial T_{11}}{\partial X} - \rho_0 \frac{\partial v_1}{\partial t} = 0, \tag{2.4}$$

$$\frac{\partial T_{1a}}{\partial X} - \rho_0 \frac{\partial v_a}{\partial t} = 0. \tag{2.5}$$

**Balance of angular momentum:**

$$\frac{\partial M_{11}}{\partial X} + R_1 - \rho_0 J_0 \left[ \frac{\partial w_1}{\partial t} - \frac{1}{2} \left( \varphi_3 \frac{\partial \psi_2}{\partial t} - \varphi_2 \frac{\partial \psi_3}{\partial t} \right) \right] = 0, \tag{2.6}$$

$$\frac{\partial M_{1a}}{\partial X} + R_a - \rho_0 J_0 \left[ \frac{\partial w_a}{\partial t} - \frac{1}{2} \left( \varphi_1 \frac{\partial \psi_2}{\partial t} - \varphi_2 \frac{\partial \psi_1}{\partial t} \right) \right] = 0, \quad \alpha \neq \beta, \tag{2.7}$$

where $R_a$ are defined as

$$R_1 = T_{33} - T_{32} + p_2 T_{13} - p_3 T_{12}, \tag{2.8}$$

$$R_a = (-1)^{\alpha} (T_{\beta \gamma} - T_{\gamma \beta} + p_\beta T_{a \gamma} - p_\gamma T_{a \beta}), \quad \alpha \neq \beta \tag{2.9}$$

and $T_{\alpha \beta}$ and $M_{\alpha \beta}$ are the first Piola-Kirchhoff stress and couple stress tensors, respectively, $\rho_0$ is the density in the reference state, $J_0$ is the inertia density and $\alpha$ and $\beta$ take the values 2 and 3 only.

For the study of weakly nonlinear waves, we use the constitutive equations derived in [6] for isotropic and homogeneous micropolar elastic media. In the derivation of the constitutive equations, first, the strain energy function is expanded into a power series in terms of the joint invariants of the strain tensors about the natural state. Then the strain tensors are also expanded into power series, and only terms up to the second degree in microrotation and the displacement and microrotation gradients are kept. Thus the constitutive equations which will be employed in this study include both the "geometrical" and "material" nonlinearities. The explicit forms of the first Piola-Kirchhoff stress tensor, $T_{\alpha \beta}$, the couple stress tensor, $M_{\alpha \beta}$, and the functions $R_{\alpha}$ are given as follows (for details, see [6]):

**Stress components:**

$$T_{11} = (\lambda + 2\mu + \kappa) p_1 + \frac{1}{2} a_1 \varphi_1^2 + \frac{1}{2} a_2 \varphi_2 \varphi_1 + a_3 (\varphi_3 p_2 - \varphi_2 p_3) + \frac{1}{2} a_4 p_1^2$$

$$+ \frac{1}{2} a_5 \varphi_2 \varphi_3 + \frac{1}{2} a_6 \psi_1^2 + \frac{1}{2} a_7 \psi_2 \psi_3, \tag{2.10}$$

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\(^1\) Unless otherwise stated, summation convention is valid for repeated indices throughout the text, and Latin indices take the values 1, 2, and 3, and Greek indices take the values 2 and 3 only.
\[ T_{1e} = (\mu + \kappa) p_e - (-1)^{s} \kappa \varphi \gamma + \frac{1}{4} (a_1 - a_2) \varphi \varphi_1 + (-1)^{s} a_3 \varphi \beta p_1 + a_5 p_1 p_e + a_6 \gamma \psi_1 \psi_e, \quad \alpha \neq \beta. \]  
(2.11)

Couple stress components:
\[ M_{11} = (\alpha + \beta + \gamma) \psi_1 + a_6 p_1 \psi_1 + a_8 \psi_4 \psi_e + \frac{1}{2} (a_{10} + a_8 - a_9) (\varphi \gamma \psi_1 - \varphi \gamma \psi_2), \]  
(2.12)
\[ M_{1e} = \gamma \psi_1 + a_7 p_1 \psi_4 + a_8 \psi_4 \psi_1 + (-1)^{s} a_6 \varphi \gamma \psi_1 + (-1)^{s} a_{10} (\varphi \beta \psi_1 - \varphi \beta \psi_e), \quad \alpha \neq \beta. \]  
(2.13)

Functions \( R_k \):
\[ R_1 = -2 \kappa \varphi_1 - a_1 \varphi_1 p_1 + (a_2 + a_3 - \frac{1}{2} a_1) \varphi_4 p_s, \]  
(2.14)
\[ R_2 = -2 \kappa \varphi_2 - (-1)^{s} \kappa p_2 - \frac{1}{2} (a_1 + 2a_3) \varphi_1 p_s - a_2 \varphi_4 p_1 + (-1)^{s} a_3 p_1 p_s - (-1)^{s} a_8 \varphi_1 \psi_e, \quad \alpha \neq \beta, \]  
(2.15)

where \( \lambda, \mu, \kappa, \alpha, \beta \) and \( \gamma \) are linear material constants and the coefficients of nonlinear terms, \( a_k (k = 1, \ldots, 10) \), are related to nonlinear material constants of the medium in the case of material nonlinearity. In the case of geometric nonlinearity, the coefficients \( a_k (k = 1, \ldots, 10) \) depend only on the linear material constants and take the following form:
\[ a_1 = -2 \lambda, \quad a_2 = -2 (\lambda + \mu - \kappa/2), \quad a_3 = \lambda + \mu, \]  
\[ a_4 = a_5 = a_6 = a_7 = a_8 = 0, \quad a_9 = \alpha + \beta, \quad a_{10} = \gamma/2. \]  
(2.16)

For further convenience, the following compatibility equations are written:
\[ \frac{\partial p_k}{\partial t} - \frac{\partial \psi_k}{\partial X} = 0, \quad \frac{\partial \psi_k}{\partial t} - \frac{\partial w_k}{\partial X} = 0, \quad \frac{\partial w_k}{\partial t} - w_k = 0. \]  
(2.17)

The asymptotic analysis in the next section will be based on the system of first-order differential equations given by equations (2.4)–(2.7) and (2.17).

2.2. Dispersion relations

In order to see the dispersive character of the micropolar elastic medium, first, the nonlinear field equations given by (2.4)–(2.15) are linearized about the natural state. Then assumed harmonic wave solutions for \( u_i \) and \( \varphi_i \) in the form \( \exp \{ i(k X - \omega t) \} \) where \( k \) and \( \omega \) denote wavenumber and circular frequency, respectively, are substituted into the linearized field equations. This gives the following dispersion relation [6]:
\[ D_1 (k, \omega) D_2 (k, \omega) [D_3 (k, \omega)]^2 = 0, \]  
(2.18)
where \( D_1, D_2 \) and \( D_3 \) are given by
\[ D_1 (k, \omega) = \omega^2 - c_l^2 k^2, \]  
\[ D_2 (k, \omega) = \omega^2 - b_l^2 k^2 - \omega_0^2, \]  
\[ D_3 (k, \omega) = 2 (\omega^2 - c_l^2 k^2) (\omega^2 - b_l^2 k^2 - \omega_0^2) - \omega_0^2 c_l^2 k^2. \]  
(2.19)

Here the following definitions are used:
\[ c_l^2 = (\lambda + 2 \mu + \kappa)/\rho_0, \quad c_t^2 = (\mu + \kappa)/\rho_0, \quad c_0^2 = \kappa/\rho_0, \]  
\[ b_l^2 = (\alpha + \beta + \gamma)/\rho_0 J_0, \quad b_t^2 = \gamma/\rho_0 J_0, \quad \omega_0^2 = 2 \kappa/\rho_0 J_0. \]  
(2.20)

In equation (2.18), \( D_1 \) and \( D_2 \) represent the dispersion relations corresponding to the longitudinal displacement and longitudinal microrotation modes associated with \( u_i \) and \( \varphi_i \), respectively. \( D_3 \), on the other hand, is the dispersion relation corresponding to the coupled transverse displacement and microrotation modes. As can be seen from equations (2.18) and (2.19), the longitudinal modes are decoupled from the transverse modes and from each other, and the longitudinal microrotation mode is strongly dispersive, whereas the corresponding displacement mode is nondispersive. This property of a linear longitudinal
microrotation wave, i.e. being strongly dispersive, may lead to an NLS equation, as shown in [6], for the slow modulation of its complex amplitude if the medium includes nonlinearity. Moreover, equations (2.18) and (2.19) show that the transverse modes are also dispersive and the amplitudes of two transverse waves (e.g. the amplitudes related to \( \varphi_2 \) and \( \varphi_3 \)) are known in terms of the remaining transverse wave amplitudes (related to \( u_2 \) and \( u_3 \)). In addition, these waves satisfy the same dispersion relation. Because of this feature of linear transverse waves, i.e. possession of an identical dispersion relation for two linearly polarized transverse waves, one may expect a wave–wave interaction between these transverse modes if nonlinearity is included in the analysis. This feature of linear transverse waves also constitutes the crucial point of the analysis in [7] where the nonlinear amplitude modulation of transverse waves is described by two coupled NLS equations.

3. INTERACTION EQUATIONS

3.1. Resonance condition

As pointed out in the previous section, the dispersion relation corresponding to transverse waves in a linear micropolar elastic solid is independent from those of longitudinal waves and is given by \( D_3(k, \omega) = 0 \). This dispersion relation has two branches, the so-called acoustical branch which involves low frequencies, \( \omega_1 \), and the optical branch which involves a range of higher frequencies, \( \omega_2 \). In this section, we examine the nonlinear interaction between transverse acoustical and optical waves in a weakly nonlinear micropolar elastic solid whose governing equations are given in Section 2. We now assume that, for the wavenumbers \( k_1 \) and \( k_2 \) corresponding to acoustical and optical branches, respectively, there exist waves having equal group velocity in both branches (see Fig. 1). Thus, the wavenumbers \( k_j \) and the frequencies \( \omega_j \) of these waves will satisfy the dispersion relation \( D_3(k_j, \omega_j) = 0 \) and the following relations:

\[
\omega_1^2 = P_1 - Q_1, \quad \omega_2^2 = P_2 + Q_2, \quad (3.1)
\]

where \( P_j \) and \( Q_j \) are given by

\[
P_j = \frac{[(b^2_j + c_j^2) k_j^2 + \omega_j^2]}{2}, \quad Q_j = \frac{[(b^2_j - c_j^2) k_j^2 + 2(b^2_j - c_j^2) \omega_j^3 + \omega_j^6 + 2 \omega_j^5 c_j k_j^2]}{1/2}. \quad (3.2)
\]

Here subscripts 1 and 2 represent acoustical and optical modes, respectively. The group velocities \( c_{gj} \) of these modes are given as follows:

\[
c_{gj} = \frac{d\omega_j}{dk_j} = \frac{k_j [2 \omega_j^2 (b^2_j + c_j^2) - 4 b_j c_j k_j^2 - 2 c_j^2 \omega_j^3 + \omega_j^6 c_j^2]}{2 \omega_j [2 \omega_j^2 - (b_j^2 + c_j^2) k_j^2 - \omega_j^6]}. \quad (3.3)
\]

In addition, the group velocities satisfy the following resonance condition:

\[
c_{g1} = c_{g2} = c_{g}. \quad (3.4)
\]

A typical graphical representation of the resonance condition is shown in Fig. 1. Since we consider transverse waves, we assume that

\[
D_1(k_j, \omega_j) \neq 0, \quad D_2(k_j, \omega_j) \neq 0. \quad (3.5)
\]

Further, to eliminate the self-resonance, we also assume that

\[
D_3(2k_j, 2\omega_j) \neq 0, \quad D_3(k_1 \mp k_2, \omega_1 \mp \omega_2) \neq 0. \quad (3.6)
\]

3.2. Interaction equations for linearly polarized waves

The reductive perturbation method [12] will be used to derive interaction equations. We now introduce the following scale transformation:

\[
\xi = \varepsilon(X - c_s t), \quad \tau = \varepsilon^2 t. \quad (3.7)
\]

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*Throughout the text, index \( j \) takes the values 1 and 2 only, and summation convention is not valid for repeated index \( j \).*
where $\xi$ and $\tau$ are slow variables, $\varepsilon$ is a small parameter measuring the strength of nonlinearity. We then assume that all the variables, i.e. $p_k, v_k, \varphi_k, \psi_k$, and $w_k (k = 1, 2, 3)$, have the following series solutions expanded in terms of small parameter $\varepsilon$ about the natural state:

$$\Theta = \varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \cdots,$$

where the first- and second-order solutions are in the form

$$\Theta_1 = [\Theta_1^{(1)}(\xi, \tau) e^{i\theta_1} + \Theta_1^{(2)}(\xi, \tau) e^{i\theta_2}] + \text{c.c.}$$

$$\Theta_2 = [\Theta_2^{(1)}(\xi, \tau) + \Theta_2^{(2)}(\xi, \tau) e^{i\theta_1} + \Theta_2^{(3)}(\xi, \tau) e^{i\theta_2} + \Theta_2^{(4)}(\xi, \tau) e^{i\theta_1 + \theta_2} + \Theta_2^{(5)}(\xi, \tau) e^{i\theta_1 - \theta_2}] + \text{c.c.}$$

Here $\Theta$ is used to represent any of the field variables, c.c. stands for the complex conjugate of the preceding term, and $\theta_j (j = 1, 2)$ are the rapidly rotating phases defined by $\theta_j = k_j \chi - \omega_j t$.

As seen from equations (3.9) and (3.10), the stretching transformation separates the solution into rapidly varying parts associated with the oscillations and slowly varying envelopes of carrier waves with these fast oscillations. The starting solution given in (3.9) includes only plane waves centered around the frequencies $\omega_j$ and the wavenumbers $k_j$. $\Theta_1^{(j)}$ are the slowly varying complex amplitudes of these waves with equal group velocity. The second-order solution given in (3.10) is written so as to include all possible terms to appear in the second-order equations. We now want to find out how the slowly varying amplitudes, $\Theta_1^{(j)}$, are affected by nonlinearity.

Using the slow variables (3.7) and the expansion (3.8) in the field equations (2.4)–(2.7) and (2.17), and then equating the coefficients of powers of $\varepsilon$ to zero, we obtain a hierarchy of equations to be satisfied for each order of $\varepsilon$. First applying this procedure to the compatibility equations (2.17), we obtain the amplitudes corresponding to $p_k, \varphi_k, \psi_k$, and $w_k$ in terms of the amplitudes corresponding to $p_k$ and $\varphi_k$. These relations are given in Appendix A for the first three orders. Using the remaining equations (2.4)–(2.7) for $O(\varepsilon)$, we find the following relations $(j = 1, 2)$:

$$D_1(k_j, \omega_j) p^{(j)}_{11} = 0, \quad D_2(k_j, \omega_j) \varphi^{(j)}_{11} = 0, \quad D_3(k_j, \omega_j) p^{(j)}_{s1} = 0$$

and

$$\varphi^{(j)}_{s1} = -(-1)^\alpha \bar{c}_j p^{(j)}_{\beta 1}, \quad \alpha \neq \beta,$$

where

$$\bar{c}_j = (c_k^2 k_j^2 - \omega_j^2) / c_k^2 k_j^2.$$

Recalling that $D_3(k_j, \omega_j) = 0, D_1(k_j, \omega_j) \neq 0$ and $D_2(k_j, \omega_j) \neq 0$, we obtain the following results for the first-order problem:

$$p^{(j)}_{11} = \varphi^{(j)}_{11} = 0, \quad \varphi^{(j)}_{21} = -\bar{c}_j \psi_j, \quad \varphi^{(j)}_{s1} = \bar{c}_j \varphi_j,$$

as expected.
where, for convenience, the amplitudes \( p_{12}^{(j)} \) which will be determined from higher-order perturbation problems are shown as follows:

\[
p_{21}^{(j)} = \Phi_j, \quad p_{31}^{(j)} = \Psi_j. \tag{3.15}
\]

As seen from these relations, all the first-order longitudinal wave amplitudes are zero, whereas the transverse microrotation wave amplitudes are known in terms of the amplitudes of two pairs of two linearly polarized transverse displacement waves, namely, \( \Phi_j \) and \( \Psi_j \) \((j = 1, 2)\).

We now consider the second-order problem and obtain the following results for each mode \((j = 1, 2)\):

\[
e^{i\theta_0}: \quad \varphi_{12}^{(0)} = 0, \quad \varphi_{22}^{(0)} = -\frac{1}{2}(-1)^\alpha \varphi_{\beta_{22}}^{(0)} \quad \alpha \neq \beta,
\]

\[
e^{i\theta_1}: \quad p_{12}^{(j)} = \varphi_{12}^{(j)} = 0, \quad \varphi_{22}^{(j)} = -\bar{c}_j \varphi_{32}^{(j)} = -\frac{2i\omega_j}{c_0^2 k_0} \left( c_s - \frac{\omega_j}{k_j} \right) \varphi_{32}^{(j)}, \quad \varphi_{32}^{(j)} = \bar{c}_j \varphi_{32}^{(j)} + \frac{2i\omega_j}{c_0^2 k_0} \left( c_s - \frac{\omega_j}{k_j} \right) \varphi_{32}^{(j)}.
\]

\[
e^{i2\theta_1}: \quad \tilde{p}_{12}^{(j)} = \varphi_{12}^{(j)} = 0, \quad \tilde{p}_{12}^{(j)} = S_j(\Phi_j^\ast + \Psi_j^\ast),
\]

\[
e^{i(\theta_1 + \theta_2)}: \quad p_{12}^{(+)} = \varphi_{12}^{(+)} = 0, \quad p_{12}^{(+)} = H_+ (\Phi_1 \Phi_2 + \Psi_1 \Psi_2),
\]

\[
e^{i(\theta_1 - \theta_2)}: \quad p_{12}^{(-)} = \varphi_{12}^{(-)} = 0, \quad p_{12}^{(-)} = H_-(\Phi_1 \Phi_2^\ast + \Psi_1 \Psi_2^\ast),
\]

\[
\varphi_{12}^{(+)} = M_+(\Phi_1 \Phi_2 + \Psi_1 \Psi_2), \quad \varphi_{12}^{(-)} = M_-(\Phi_1 \Phi_2^\ast + \Psi_1 \Psi_2^\ast),
\]

\[
(3.16)
\]

where the asterisk denotes complex conjugate, \( S_j, H_\pm \) and \( M_\pm \) are real functions of wavenumbers and material parameters, and their explicit forms are given in Appendix B. Note that all the second-order longitudinal wave amplitudes corresponding to the modes \( e^{i\theta_0} \) are zero, and that the amplitudes \( p_{12}^{(j)} \) are arbitrary at this order of the analysis. It is also interesting to note that, at this order of the approximation, the amplitudes corresponding to transverse directions are zero for the modes \( e^{i2\theta_1} \) and \( e^{i(\theta_1 + \theta_2)} \) and the interaction between acoustical and optical modes arise through the longitudinal components (see \( p_{12}^{(+)} \) and \( \varphi_{12}^{(+)} \)). The reason for this situation is that the longitudinal components are, at least, at \( O(\varepsilon^2) \) due to equations (3.14) and consequently the nonlinear terms in the field equations corresponding to the transverse directions are, at least, at \( O(\varepsilon^3) \) (see equations (2.11), (2.13) and (2.15)).

We now proceed to the third-order problem to determine the behavior of the complex amplitudes \( \Phi_j \) and \( \Psi_j \). We will not present here all the third-order equations as the remaining equations are not needed. From the relations corresponding to the zeroth mode, we find that

\[
p_{12}^{(0)} = G_1(|\Phi_1|^2 + |\Psi_1|^2) + G_2(|\Phi_2|^2 + |\Psi_2|^2), \tag{3.17}
\]

where \( G_j \) are real functions of wavenumbers and material parameters, and their explicit forms are given in Appendix B. The relations corresponding to the modes \( e^{i\theta_0} \), will give some nonsecularity conditions for the amplitudes \( \Phi_j \) and \( \Psi_j \). Since the explicit forms of these equations are too complicated, only the resulting equations obtained after eliminating higher-order terms will be given here. After lengthy calculations, they are in the form:

\[
i \frac{\partial \Phi_1}{\partial t} + \Gamma_1 \frac{\partial^2 \Phi_1}{\partial \xi^2} + \Delta_{11} \Phi_1 (|\Phi_1|^2 + |\Psi_1|^2) + \Delta_{12} \Phi_1 (\Phi_1^\ast + \Psi_1^\ast)
\]

\[
+ \Phi_1 (\Lambda_{11} |\Phi_2|^2 + \Lambda_{12} |\Psi_2|^2) + \Psi_1 (\Lambda_{13} \Phi_2 \Psi_2^\ast + \Lambda_{14} \Phi_2^\ast \Psi_2) = 0, \quad \Delta_{11} = \Phi_1 (\Lambda_{11} |\Phi_2|^2 + \Lambda_{12} |\Psi_2|^2) + \Psi_1 (\Lambda_{13} \Phi_2 \Psi_2^\ast + \Lambda_{14} \Phi_2^\ast \Psi_2) = 0,
\]

\[
i \frac{\partial \Psi_1}{\partial t} + \Gamma_1 \frac{\partial^2 \Psi_1}{\partial \xi^2} + \Delta_{11} \Psi_1 (|\Phi_1|^2 + |\Psi_1|^2) + \Delta_{12} \Psi_1 (\Phi_1^\ast + \Psi_1^\ast)
\]

\[
+ \Psi_1 (\Lambda_{11} |\Psi_2|^2 + \Lambda_{12} |\Phi_2|^2) + \Phi_1 (\Lambda_{13} \Phi_2 \Psi_2^\ast + \Lambda_{14} \Phi_2^\ast \Psi_2) = 0,
\]
\[
\begin{align*}
&\frac{i}{\partial t} \frac{\partial \Phi_2}{\partial x} + \frac{\partial^2 \Phi_2}{\partial x^2} + \Delta_{21} \Phi_2 (|\Phi_2|^2 + |\Psi_1|^2) + \Delta_{22} \Phi_2^* (\Phi_2^2 + \Psi_2^2) \\
&\quad + \Phi_2 (\Delta_{21} |\Phi_1|^2 + \Delta_{22} |\Psi_1|^2) + \Psi_2 (\Delta_{23} \Phi_1 \Psi_1^* + \Delta_{24} \Phi_1^* \Psi_1) = 0, \\
&\frac{i}{\partial t} \frac{\partial \Psi_2}{\partial x} + \frac{\partial^2 \Psi_2}{\partial x^2} + \Delta_{21} \Psi_2 (|\Phi_2|^2 + |\Psi_2|^2) + \Delta_{22} \Psi_2^* (\Phi_2^2 + \Psi_2^2) \\
&\quad + \Psi_2 (\Delta_{21} |\Psi_1|^2 + \Delta_{22} |\Phi_1|^2) + \Phi_2 (\Delta_{23} \Psi_1^* \Phi_1 + \Delta_{24} \Phi_1^* \Psi_1) = 0,
\end{align*}
\]
where the coefficients $\Gamma_j, \Delta_{j1}, \Delta_{j2}, \Lambda_j, \Lambda_{j3}$ and $\Lambda_{j4}$ ($j = 1, 2$) are real functions of wavenumbers and material parameters, and their explicit forms are given in Appendix B. The evolution equations (3.18) describe the variation of the complex amplitudes of the transverse acoustical and optical waves with linear polarization along the $X_{2}^\perp$- and $X_3^\perp$-directions. These evolution equations are also valid, with slightly different coefficients, for the amplitudes of the transverse microwavation waves, $\phi_{\alpha}^{(1)}$ ($\alpha = 2, 3, j = 1, 2$), because of the relations (3.14). From Appendix B, it should also be noted that the coefficients $S_{j}$, $H_{\perp}$, $M_{\perp}$ and $G_{j}$ become infinitely large when $D_{1}(2k_{j}, 2\omega_{j}) = 0$, $D_{1}(k_{1} \mp k_{2}, \omega_{1} \pm \omega_{2}) = 0$, and $D_{1}(k_{1} \mp k_{2}, \omega_{1} \pm \omega_{2}) = 0$ and $c_{p} = c_{s}$, respectively. The coefficients $\Delta_{j1}, \Delta_{j2}, \Lambda_{j1}, \Lambda_{j2}, \Lambda_{j3}$ and $\Lambda_{j4}$ then become infinitely large and the interaction equations (3.18) break down. To avoid such a case, we further require that these conditions are not satisfied by the pairs $(k_{j}, \omega_{j})$.

3.3. Interaction equations for circularly polarized waves

Recalling equations (3.8) and (3.9), we see that the displacement gradient vector in the first-order approximation takes the form:
\[
p = \varepsilon(\Phi_{1} e^{i\theta_{1}} + \Phi_{2} e^{i\theta_{2}}) + c.c.] I_{2} + \varepsilon(\Psi_{1} e^{i\theta_{1}} + \Psi_{2} e^{i\theta_{2}}) + c.c.] I_{3} + O(\varepsilon^{2}),
\]
where $I_{2}$ and $I_{3}$ are unit vectors in the $X_{2}^\perp$- and $X_3^\perp$-directions, respectively. It should be noted that although the longitudinal displacement gradient is zero in the first order of the approximation, in general, it is different from zero in the second order as a result of nonlinear self- and mutual-interactions of transverse waves. To examine the properties of the solutions, it is more convenient to rewrite equations (3.18) in terms of the complex amplitudes of pairs of two circularly polarized transverse waves. With this aim, we make the following transformation:
\[
\phi_{j} = (\Phi_{j} - i\Psi_{j})/\sqrt{2}, \quad \psi_{j} = (\Phi_{j} + i\Psi_{j})/\sqrt{2},
\]
where $|\Phi_{j}|^2 + |\Psi_{j}|^2 = |\phi_{j}|^2 + |\psi_{j}|^2$, and $\phi_{j}$ and $\psi_{j}$ represent the complex amplitudes of the circularly polarized waves. The displacement gradient vector in the first-order approximation then takes the form:
\[
p = \varepsilon(p_{+} I_{+} + p_{-} I_{-}) + O(\varepsilon^{2}),
\]
where $p_{\pm}$ and $I_{\pm}$ are defined by
\[
p_{\pm} = \phi_{1} e^{i\theta_{1}} + \phi_{2} e^{i\theta_{2}} + \psi_{1} e^{-i\theta_{1}} + \psi_{2} e^{-i\theta_{2}}, \quad p_{-} = \psi_{1} e^{i\theta_{1}} + \psi_{2} e^{i\theta_{2}} + \phi_{1} e^{-i\theta_{1}} + \phi_{2} e^{-i\theta_{2}},
\]
and
\[
I_{+} = (I_{2} + iI_{3})/\sqrt{2}, \quad I_{-} = (I_{2} - iI_{3})/\sqrt{2},
\]
respectively. Thus, by making use of this transformation in equations (3.18), we reach the following four coupled nonlinear evolution equations:
\[
\begin{align*}
&i \frac{\partial \phi_{1}}{\partial t} + \Gamma_{1} \frac{\partial^{2} \phi_{1}}{\partial \xi^{2}} + [\Delta_{11} |\phi_{1}|^2 + (\Delta_{11} + 2\Delta_{12}) |\psi_{1}|^2 + \nu_{11} |\phi_{2}|^2 + \nu_{12} |\psi_{2}|^2] \phi_{1} \\
&\quad + (\nu_{13} \phi_{2} \psi_{3} + \nu_{14} \phi_{3} \psi_{2}) \psi_{1} = 0, \\
&i \frac{\partial \psi_{1}}{\partial t} + \Gamma_{1} \frac{\partial^{2} \psi_{1}}{\partial \xi^{2}} + [\Delta_{11} |\psi_{1}|^2 + (\Delta_{11} + 2\Delta_{12}) |\phi_{1}|^2 + \nu_{11} |\psi_{2}|^2 + \nu_{12} |\phi_{2}|^2] \psi_{1} \\
&\quad + (\nu_{13} \phi_{2} \psi_{3} + \nu_{14} \phi_{3} \psi_{2}) \phi_{1} = 0,
\end{align*}
\]
\[
\begin{align*}
\frac{i}{\tau} \frac{\partial \phi_2}{\partial \tau} + \Gamma_2 \frac{\partial^2 \phi_2}{\partial \xi^2} + [\Delta_{21} \phi_2^2] + (\Delta_{21} + 2\Delta_{22}) \psi_2^2 + v_{21} \phi_1^2 + v_{22} \phi_1^2 \phi_2^2 & = 0, \\
(\nu_{23} \phi_1 \psi_1 + \nu_{24} \phi_1 \psi_1 \psi_1^* \psi_1) & = 0, \\
\frac{i}{\tau} \frac{\partial \psi_2}{\partial \tau} + \Gamma_2 \frac{\partial^2 \psi_2}{\partial \xi^2} + [\Delta_{21} \psi_2^2] + (\Delta_{21} + 2\Delta_{22}) \phi_2^2 + v_{21} \phi_1^2 + v_{22} \phi_1^2 \phi_2^2 & = 0, \\
(\nu_{23} \phi_1^* \psi_1 + \nu_{24} \phi_1^* \psi_1 \psi_1^* \phi_2^2 & = 0, \\
(3.24)
\end{align*}
\]

where, for \( j = 1, 2 \),
\[
\begin{align*}
\nu_{j1} = (\Lambda_{j1} + \Lambda_{j2} + \Lambda_{j3} - \Lambda_{j4})/2, \\
\nu_{j2} = (\Lambda_{j1} + \Lambda_{j2} - \Lambda_{j3} + \Lambda_{j4})/2, \\
\nu_{j3} = (\Lambda_{j1} - \Lambda_{j2} + \Lambda_{j3} + \Lambda_{j4})/2, \\
\nu_{j4} = (\Lambda_{j1} - \Lambda_{j2} - \Lambda_{j3} - \Lambda_{j4})/2. \\
(3.25)
\end{align*}
\]

Similar coupled evolution equations with different forms of the last terms have been obtained in many branches of physics and engineering, for instance, in surface gravity waves [13], nonlinear optics [14] and so on [15]. For more references on these equations which are called four-wave interaction equations, the reader is referred to [12, 16–18].

In equations (3.24) the second terms represent the dispersive effect and the nonlinear terms are of two types, i.e. the terms inside the brackets and the last terms. The first terms inside the brackets represent the interaction of a wave with itself. The other terms inside the brackets describe the mutual interaction of pairs of waves. The last terms describe the energy transfer among the different components.

3.4. Energy conservation

In order to see how the last terms in equations (3.24) are related to the energy transfer, we first write the following integrals by multiplying equations (3.24) by \( \phi_1^*, \psi_1^*, \phi_2^* \) and \( \psi_2^* \), respectively, and subtracting the complex conjugates
\[
\begin{align*}
\frac{1}{i\tau} \int_{-\infty}^{+\infty} |\phi_1|^2 d\xi = v_{13} J_3 + v_{14} J_4, \\
\frac{1}{i\tau} \int_{-\infty}^{+\infty} |\psi_1|^2 d\xi = - v_{13} J_3 - v_{14} J_4, \\
\frac{1}{i\tau} \int_{-\infty}^{+\infty} |\phi_2|^2 d\xi = - v_{23} J_3 + v_{24} J_4, \\
\frac{1}{i\tau} \int_{-\infty}^{+\infty} |\psi_2|^2 d\xi = v_{23} J_3 - v_{24} J_4, \\
(3.26)
\end{align*}
\]

where \( J_3 \) and \( J_4 \) are defined by
\[
\begin{align*}
J_3 = \int_{-\infty}^{+\infty} (\phi_1 \psi_1^* \phi_2^* \psi_2 - \phi_1^* \psi_1 \phi_2 \psi_2^*) d\xi, \\
J_4 = \int_{-\infty}^{+\infty} (\phi_1 \psi_1^* \phi_2 \psi_2^* - \phi_1^* \psi_1 \phi_2 \psi_2^*) d\xi. \\
(3.27)
\end{align*}
\]

Here we have assumed that \( \phi_j \) and \( \psi_j \) (\( j = 1, 2 \)) decrease sufficiently rapidly for arbitrarily large \(|\xi|\). Note that the terms inside the parentheses in equations (3.27) and consequently \( J_3 \) and \( J_4 \) are purely imaginary. As can be seen from equations (3.26), each wave conserves its energy if \( \nu_{j3} = \nu_{j4} = 0 \), i.e. the last terms in equations (3.24) vanish. Moreover, these equations show that the total energies corresponding to the acoustical and optical modes are conserved separately (\( j = 1, 2 \)),
\[
\frac{d}{d\tau} \int_{-\infty}^{+\infty} (|\phi_j|^2 + |\psi_j|^2) d\xi = 0
\]
\[
(3.28)
\]
and consequently the total energy of the system is conserved
\[
\frac{d}{d\tau} \int_{-\infty}^{+\infty} (|\phi_1|^2 + |\psi_1|^2 + |\phi_2|^2 + |\psi_2|^2) d\xi = 0
\]
\[
(3.29)
\]
from which all the solutions are found to be bounded if they exist.
3.5. Two coupled NLS equations

For some special cases, equations (3.24) reduce to the two coupled NLS equations

\[
\begin{align*}
1 & \frac{\partial f_1}{\partial t} + \gamma_{1} \frac{\partial^2 f_1}{\partial \xi^2} + (\alpha_1 |f_1|^2 + \beta_1 |f_2|^2) f_1 = 0, \\
1 & \frac{\partial f_2}{\partial t} + \gamma_{2} \frac{\partial^2 f_2}{\partial \xi^2} + (\alpha_2 |f_2|^2 + \beta_2 |f_1|^2) f_2 = 0.
\end{align*}
\]  

(3.30)

We now present a list of the complex amplitudes \(f_j\), and the coefficients \(\alpha_j, \beta_j\) and \(\gamma_j\) for various special cases.

**Case (i):** There exist waves corresponding to the acoustical (or optical) branch only in the medium. Thus we have \(\Phi_2 = \Psi_2 = 0\) (or \(\Phi_1 = \Psi_1 = 0\)) and consequently \(\phi_2 = \psi_2 = 0\) (or \(\phi_1 = \psi_1 = 0\)). Then we find that

\[
f_1 = \phi_j, \quad f_2 = \psi_j, \quad \gamma_1 = \gamma_2 = \Gamma_j, \quad \alpha_1 = \alpha_2 = \Delta_{j1}, \quad \beta_1 = \beta_2 = \Delta_{j1} + 2\Delta_{j2},
\]

where \(j = 1\) (or \(j = 2\)) corresponds to the case in which acoustical (or optical) waves exist in the medium only.

**Case (ii):** Both acoustical and optical waves are linearly polarized along the \(X_2^+\) (or \(X_2^-\)) direction. Thus we have \(\Psi_1 = 0\) (or \(\Phi_1 = 0\)) and consequently \(\phi_j = \psi_j\) (or \(\phi_j = -\psi_j\)). Then we find that

\[
f_1 = \phi_1, \quad f_2 = \phi_2, \quad \gamma_j = \Gamma_j, \quad \alpha_j = 2(\Delta_{j1} + \Delta_{j2}), \quad \beta_j = \nu_{j1} + \nu_{j2} + \nu_{j3} + \nu_{j4}.
\]

(3.32)

**Case (iii):** Acoustical and optical waves are linearly polarized along the \(X_2^+\) and \(X_2^-\) (or \(X_3^+\) and \(X_3^-\)) directions, respectively. Thus we have \(\Psi_1 = \Phi_2 = 0\) (or \(\Psi_2 = \Phi_1 = 0\)) and consequently \(\phi_1 = \psi_1\) and \(\phi_2 = -\psi_2\) (or \(\phi_1 = -\psi_1\) and \(\phi_2 = \psi_2\)). Then we find that

\[
f_1 = \phi_1, \quad f_2 = \phi_2, \quad \gamma_j = \Gamma_j, \quad \alpha_j = 2(\Delta_{j1} + \Delta_{j2}), \quad \beta_j = \nu_{j1} + \nu_{j2} - \nu_{j3} - \nu_{j4}.
\]

(3.33)

**Case (iv):** There exist right, i.e. clockwise, circularly polarized waves only in the medium. Thus we have \(\psi_j = 0\). Then we find that

\[
f_1 = \phi_1, \quad f_2 = \phi_2, \quad \gamma_j = \Gamma_j, \quad \alpha_j = \Delta_{j1}, \quad \beta_j = \nu_{j1}.
\]

(3.34)

**Case (v):** There exist left, i.e. counterclockwise, circularly polarized waves only in the medium. Thus we have \(\phi_j = 0\). Then we find that

\[
f_1 = \psi_1, \quad f_2 = \psi_2, \quad \gamma_j = \Gamma_j, \quad \alpha_j = \Delta_{j1}, \quad \beta_j = \nu_{j1}.
\]

(3.35)

4. SOME SPECIAL SOLUTIONS

The four-wave interaction equations given by equations (3.24) have nonlinear plane wave and envelope solitary wave solutions as the travelling-wave solutions provided that

\[
\Delta_{j2} = 0, \quad \nu_{j1} = \nu_{j2}, \quad \nu_{j3} = -\nu_{j4},
\]

(4.1)

for which the evolution equations take the following form:

\[
\begin{align*}
1 & \frac{\partial \phi_1}{\partial t} + \Gamma_1 \frac{\partial^2 \phi_1}{\partial \xi^2} + [\Delta_{11} (|\phi_1|^2 + |\psi_1|^2) + \nu_{11} (|\phi_2|^2 + |\psi_2|^2)] \phi_1 \\
& \quad + \nu_{13} (\phi_2 \psi_2 - \phi_2^* \psi_2) \psi_1 = 0, \\
1 & \frac{\partial \psi_1}{\partial t} + \Gamma_1 \frac{\partial^2 \psi_1}{\partial \xi^2} + [\Delta_{11} (|\psi_1|^2 + |\phi_1|^2) + \nu_{11} (|\phi_2|^2 + |\psi_2|^2)] \psi_1 \\
& \quad - \nu_{13} (\phi_2 \psi_2 - \phi_2^* \psi_2) \phi_1 = 0,
\end{align*}
\]
\[
\frac{i}{2} \frac{\partial \phi_2}{\partial t} + \Gamma_2 \frac{\partial^2 \phi_2}{\partial \xi^2} + [\Delta_{21} (|\phi_2|^2 + |\psi_2|^2) + \nu_{22} (|\phi_1|^2 + |\psi_1|^2)] \phi_2
\]
\[
+ \nu_{23} (\phi_1 \psi_1^* - \phi_1^* \psi_1) \psi_2 = 0,
\]
\[
\frac{i}{2} \frac{\partial \psi_2}{\partial t} + \Gamma_2 \frac{\partial^2 \psi_2}{\partial \xi^2} + [\Delta_{21} (|\phi_2|^2 + |\psi_2|^2) + \nu_{22} (|\phi_1|^2 + |\psi_1|^2)] \psi_2
\]
\[
- \nu_{23} (\phi_1 \psi_1^* - \phi_1^* \psi_1) \phi_2 = 0.
\]
\[
(4.2)
\]

Note that the terms inside the parentheses in the last terms of these equations are purely imaginary, whereas the terms inside the brackets are real.

Now let us seek the traveling wave solutions of equations (4.2) in the form:

\[
\phi_j(\xi, t) = U_j(\xi) \exp[i(K_j \xi - \Omega_j t)], \quad \psi_j(\xi, t) = V_j(\xi) \exp[i(K_j \xi - \Omega_j t)],
\]
\[
(4.3)
\]

where \( \xi = \xi - \nu_0 t \), \( U_j \) and \( V_j \) are real functions of \( \xi \), and \( K_j, \Omega_j \) and \( \nu_0 \) are real constants. If the solutions (4.3) are substituted into equations (3.19) using equation (3.20), the displacement gradient components \( p_2 \) and \( p_3 \) are found in the first-order approximation in terms of the original variables as

\[
p_2 = \sqrt{2} \epsilon [(U_1 + V_1) \cos(\tilde{k}_1 X - \tilde{\omega}_1 t) + (U_2 + V_2) \cos(\tilde{k}_2 X - \tilde{\omega}_2 t)],
\]
\[
p_3 = -\sqrt{2} \epsilon [(U_1 - V_1) \sin(\tilde{k}_1 X - \tilde{\omega}_1 t) + (U_2 - V_2) \sin(\tilde{k}_2 X - \tilde{\omega}_2 t)],
\]
\[
(4.4)
\]

where \( U_j \) and \( V_j \) are functions of \( \xi = \epsilon(X - \nu_0 t) \), and

\[
\tilde{k}_j = k_j + \epsilon K_j, \quad \tilde{\omega}_j = \omega_j + \epsilon \alpha K_j + \epsilon^2 \Omega_j, \quad \nu_0 = \epsilon \nu_0.
\]
\[
(4.5)
\]

Introducing the traveling wave solutions (4.3) into equations (4.2), the four coupled nonlinear ordinary differential equations are obtained for \( U_j \) and \( V_j \). In general, the solution can be obtained in terms of Jacobian elliptic functions under the assumption of \( \nu_0 = 2\Gamma_1 K_1 = 2\Gamma_3 K_2 \). The explicit functional forms of these solutions will not be given here, except for the following limit cases.

4.1. Uniform wave trains

In the case of uniform wave trains, the amplitudes, \( U_j \) and \( V_j \), of the traveling wave solutions (4.3) are given by

\[
U_j = A_j, \quad V_j = B_j,
\]
\[
(4.6)
\]

provided that

\[
\Omega_1 = \Gamma_1 K_1^2 - \Delta_{11} (A_1^2 + B_1^2) - \nu_{11} (A_1^4 + B_1^4),
\]
\[
\Omega_2 = \Gamma_2 K_2^2 - \Delta_{21} (A_2^2 + B_2^2) - \nu_{21} (A_2^4 + B_2^4).
\]
\[
(4.7)
\]

Here \( A_j \) and \( B_j \) are real constants. These relations show that the terms inside the brackets in equations (4.2) cause some modifications to the wave frequencies and they modify the linear dispersion relations by amplitude dispersion. Thus the solutions (4.4) together with equations (4.6) and (4.7) represent wave trains with wave speeds dependent on the constant amplitudes \( A_j \) and \( B_j \) and show how nonlinearity gives rise to interaction between elliptically polarized waves with wave speeds \( \tilde{\omega}_{1j}/\tilde{k}_1 \) and \( \tilde{\omega}_{2j}/\tilde{k}_2 \).

4.2. Envelope solitary waves

For the envelope solitary wave solutions of equations (4.2), the amplitudes \( U_j \) and \( V_j \) are given by

\[
U_j = A_j \text{ sech} C \xi, \quad V_j = B_j \text{ sech} C \xi,
\]
\[
(4.8)
\]

provided that

\[
\Omega_1 = \Gamma_1 (K_1^2 - C^2), \quad \Omega_2 = \Gamma_2 (K_2^2 - C^2).
\]
\[
(4.9)
\]
and

\[ C^2 = \frac{[\Delta_{11}(A_1^2 + B_1^2) + \nu_{11}(A_2^2 + B_2^2)]}{2\Gamma_1}, \]
\[ = \frac{[\Delta_{21}(A_3^2 + B_3^2) + \nu_{21}(A_4^2 + B_4^2)]}{2\Gamma_2}. \] (4.10)

Here \( A_j \) and \( B_j \) are real constants. In this case, equations (4.4) represent two elliptically polarized transverse waves whose envelopes are solitary waves.

We should point out that the four-wave interaction equations given by (4.2) also have a set of solutions consisting of phase jumps. In this case the amplitudes are given by

\[ U_j = A_j \tanh C_\xi, \quad V_j = B_j \tanh C_\xi \] (4.11)

provided that

\[ \Omega_1 = \Gamma_1(K_1^2 + 2C^2), \quad \Omega_2 = \Gamma_2(K_2^2 + 2C^2) \] (4.12)

and

\[ C^2 = -\frac{[\Delta_{11}(A_1^2 + B_1^2) + \nu_{11}(A_2^2 + B_2^2)]}{2\Gamma_1}, \]
\[ = -\frac{[\Delta_{21}(A_3^2 + B_3^2) + \nu_{21}(A_4^2 + B_4^2)]}{2\Gamma_2}. \] (4.13)

Here \( A_j \) and \( B_j \) are real constants. The solutions (4.4) now show how nonlinearity gives rise to interaction between elliptically polarized waves when these waves have "tanh envelopes".

As a final remark of this section, we should point out that the solutions given in equations (4.4) take the following forms:

\[ p_2 = 2\sqrt{2}\varepsilon[U_1 \cos(\tilde{k}_1 X - \tilde{\omega}_1 t) + U_2 \cos(\tilde{k}_2 X - \tilde{\omega}_2 t)], \quad p_3 = 0 \] (4.14)

when \( U_j = V_j \), and

\[ p_2 = 0, \quad p_3 = -2\sqrt{2}\varepsilon[U_1 \sin(\tilde{k}_1 X - \tilde{\omega}_1 t) + U_2 \sin(\tilde{k}_2 X - \tilde{\omega}_2 t)], \] (4.15)

when \( U_j = -V_j \). Equations (4.14) and (4.15) represent transverse nonlinear waves with linear polarization along the \( X_2 \)- and \( X_3 \)-directions, respectively. The amplitudes \( U_j \) appearing in these equations are given by equations (4.6),(4.8) or (4.11) which correspond to nonlinear plane wave, envelope solitary wave or phase jump solutions, respectively.

5. CONCLUSIONS

In this paper, we have considered the nonlinear interaction between modulated transverse acoustical and optical waves propagating in an infinite micropolar elastic medium and showed that the interaction can be described by four coupled nonlinear evolution equations in terms of the amplitudes of two pairs of two linearly or circularly polarized transverse displacement waves. A similar system of coupled evolution equations is also valid for the amplitudes of the transverse microrotation waves.

There exist two crucial points in the present analysis. The first one is that the linearized transverse waves have two independent amplitudes and that these waves have an identical dispersion relation. Note that this condition may be satisfied, in general, if the linearized system for a vector field is symmetric with respect to the coordinate axis along which the waves propagate. The second one is that the dispersion relation has two branches, acoustical and optical branches, and that the transverse waves belonging to these two branches have the same group velocity. Thus there exist four interacting waves with equal group velocity in the medium.

We also point out that these coupled nonlinear interaction equations reduce to a system of four coupled NLS equations in the absence of the last terms of the equations, which describe the energy transfer among different components. We point out that the total energies for the acoustical and optical waves are conserved separately, but the energy for each wave is not conserved in general and is transferred to others.

It should also be noted that it is not possible to derive the above four-wave interaction equations considering pure transverse motions, i.e. when \( u_1 = \varphi_1 = 0 \). In this case, the four-wave interaction equations reduce to the linear uncoupled equations as all the nonlinear terms representing the self- and mutual-interactions in these equations arise
through the second-order longitudinal displacement gradients and microrotations \( \beta_{12}^{(0)}, \beta_{12}^{(1)}, \alpha_{12}^{(2)} \) and \( \beta_{12}^{(2)} \). The reason for this situation is that the nonlinear terms in the transverse components of the stress and couple stress tensors drop in the case of pure transverse motions.

Finally, it should also be pointed out that in the previous sections a qualitative discussion of four-wave interactions has been given for a general nonlinear micropolar elastic medium and the possible existence of nonlinear wave interactions, with a special emphasis on nonlinear plane wave and solitary wave solutions, has been established. The applicability of the present solutions to real elastic crystals will form a topic for further research.

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REFERENCES


APPENDIX A

The following relations are obtained from the compatibility equations (2.17) for the first three orders:

for \( O(\psi) \)

\[
e^{i\theta_{1}}: \psi_{k1}^{(1)} = -i\frac{\psi_{k1}^{(1)}}{\psi_{k1}^{(1)}} \quad \psi_{k1}^{(1)} = ik_{1}\psi_{k1}^{(1)} \quad \psi_{k1}^{(1)} = -i\omega_{1}\psi_{k1}^{(1)}
\]

for \( O(\psi^{2}) \)

\[
e^{i\theta_{2}}: \psi_{k2}^{(2)} = 0
\]

\[
e^{i\theta_{3}}: \psi_{k3}^{(3)} = \frac{i}{2}\psi_{k1}^{(1)} \psi_{k1}^{(1)} + \frac{i}{2}\psi_{k1}^{(1)} \psi_{k1}^{(1)}
\]

\[
e^{i\theta_{4}}: \psi_{k4}^{(4)} = 2\psi_{k1}^{(1)} \psi_{k1}^{(1)} \psi_{k1}^{(1)} = \frac{2i\omega_{1}\psi_{k1}^{(1)}}{k_{1} \psi_{k1}^{(1)}}
\]

\[
e^{i(\theta_{5} + \theta_{6})}: \psi_{k6}^{(6)} = \frac{\psi_{k1}^{(1)} \psi_{k1}^{(1)} + \psi_{k1}^{(1)} \psi_{k1}^{(1)}}{k_{1} + k_{2}} = i(k_{1} + k_{2})\psi_{k1}^{(1)} \psi_{k1}^{(1)}
\]

\[
e^{i(\theta_{7} - \theta_{8})}: \psi_{k7}^{(1)} = \frac{\psi_{k1}^{(1)} - \psi_{k1}^{(1)}}{k_{1} - k_{2}} \psi_{k1}^{(1)} \psi_{k1}^{(1)}
\]

\[
e^{i(\theta_{9} - \theta_{10})}: \psi_{k10}^{(10)} = \frac{\psi_{k1}^{(1)} - \psi_{k1}^{(1)}}{k_{1} - k_{2}} \psi_{k1}^{(1)} \psi_{k1}^{(1)}
\]
for $O(\epsilon^3)$
\[
\begin{align*}
\epsilon^0: & \quad \epsilon_2^0 = -\epsilon_3^0 k_1, \quad \epsilon_3^0 = -\epsilon_3^0 k_2, \quad \epsilon_3^0 = -\epsilon_3^0 k_3, \\
\epsilon^{(1)}: & \quad \epsilon_3^{(1)} = -\frac{\Omega_0}{k_3} \epsilon_3^{(1)} + \frac{i}{k_3} (\epsilon_3 - \omega_0), \quad \epsilon_3^{(1)} = -\frac{\Omega_0}{k_3} \epsilon_3^{(1)} + \frac{i}{k_3} (\epsilon_3 - \omega_0), \\
\epsilon_3^{(1)} = & \quad \frac{\Omega_0}{k_3} \epsilon_3^{(1)} - \frac{\Omega_0}{k_3} \epsilon_3^{(1)} + \frac{i}{k_3} (\epsilon_3 - \omega_0), \quad \epsilon_3^{(1)} = -\frac{\Omega_0}{k_3} \epsilon_3^{(1)} + \frac{i}{k_3} (\epsilon_3 - \omega_0), \\
\psi_3^{(1)} = & \quad i k_3 \epsilon_3^{(1)} + \frac{\partial \varphi_3^{(1)}}{\partial \epsilon_3}, \quad \psi_3^{(1)} = -i k_3 \epsilon_3^{(1)} + \frac{\partial \varphi_3^{(1)}}{\partial \epsilon_3}.
\end{align*}
\]

APPENDIX B

The coefficient functions appearing in the second- and third-order problems:

for $O(\epsilon^4)$
\[
\begin{align*}
S_\epsilon = & \quad 2k_1^2 [\epsilon_3^2 (a_0 - \alpha_k k_1^2) + 2a_3 \vec{c}_\epsilon + a_3]/\rho_0 D_1 (2k_1, 2a_0), \\
H_\epsilon = & \quad (\kappa_1 \mp k_3)^2 \{\kappa_1 \vec{c}_\epsilon \vec{c}_3 (a_0 - \alpha_k k_1^2) + a_3 (\vec{c}_1 + \vec{c}_2) + a_3\}/\rho_0 D_1 (\kappa_1 \mp k_3, \alpha_0 \mp \alpha_0), \\
M_\epsilon = & \quad [(2a_0 \kappa_0 - a_0 - a_0) \vec{c}_\epsilon \vec{c}_3 (k_1^2 - k_1^2) - 2a_0 (\kappa_1 \mp k_3) (\vec{c}_1 \kappa_3 \pm \vec{c}_1 \kappa_3) \\
& - \rho_0 F_3 (\omega_1^2 - \omega_2^2) \vec{c}_1 \vec{c}_2 + (2a_3 + 2a_3 - a_1) (\vec{c}_1 - \vec{c}_1)])/2 \rho_0 F_3 D_2 (\kappa_1 \mp k_3, \alpha_0 \mp \alpha_0),
\end{align*}
\]

for $O(\epsilon^3)$
\[
\begin{align*}
G_\epsilon = & \quad [\epsilon_3^2 (a_2 + \alpha_k k_3^2) + 2a_3 \vec{c}_\epsilon + a_3]/\rho_0 (\epsilon_0 - \epsilon_3), \\
\Gamma_\epsilon = & \quad \frac{\rho_0}{\omega_0 k_1^2} \left[ -b_1^2 \epsilon_3^2 + \frac{b_1^2 \omega_0}{k_1} (4 \epsilon_3 - \frac{3 \omega_0}{k_1}) + \epsilon_3^2 \left( \frac{\epsilon_3 - \omega_0}{k_1} \right) \right. \\
& \left. - 4 \epsilon_3 \omega_0^2 \left( \frac{\epsilon_3 - \omega_0}{k_1} \right) + \left( \epsilon_3 - \omega_0 \right) \left( \epsilon_3 - \omega_0 \right) \right], \\
\Delta_\epsilon = & \quad k_2^2 \rho_0 (\epsilon_0^3 - \epsilon_1^3) G_\epsilon \Delta_0, \quad \Delta_\epsilon = \rho_0 (2k_1, 2a_0) S_\epsilon \Delta_0, \\
\Lambda_\epsilon = & \quad \frac{k_2^2 [\rho_0 (\epsilon_0^3 - \epsilon_1^3) G_\epsilon + R \cdot H_2^2 + R \cdot H_2^2]}{\Delta_0}, \quad \Lambda_\epsilon = \frac{k_2^2 [\rho_0 (\epsilon_0^3 - \epsilon_1^3) G_\epsilon + R \cdot H_2^2 + R \cdot H_2^2]}{\Delta_0}, \\
\Lambda_\epsilon = & \quad \Lambda_\epsilon = \frac{k_2^2 [R \cdot H_2^2 + M \cdot N^{(1)}]}{\Delta_0}, \quad \Lambda_\epsilon \Delta_\epsilon = \frac{k_2^2 [R \cdot H_2^2 + M \cdot N^{(1)}]}{\Delta_0}.
\end{align*}
\]

Here $\Delta_0, R_\epsilon$ and $N^{(1)}_\epsilon$ are given as
\[
\begin{align*}
\Delta_\epsilon = & \quad \frac{-2 \rho_0 \omega_0}{\omega_0^2 \epsilon_0^3 k_1^2} [a_0^2 \epsilon_0 \epsilon_3^2 k_1^2 + 2 (\epsilon_3^2 k_1^2 - \omega_0^2 k_1^2)], \\
R_\epsilon = & \quad \rho_0 D_1 (\kappa_1 \mp k_3, \alpha_0 \mp \alpha_0) (\kappa_1 \mp k_3)^2, \\
N^{(1)}_\epsilon = & \quad \vec{c}_1 \vec{c}_2 \left[ -\frac{\rho_0 F_0}{2} \omega_0 (a_0 \mp 2a_0) + a_0 k_1 (\kappa_1 \mp k_2) + (\kappa_1 \mp k_2) (a_0 k_1 \mp a_0 k_2) \right] \\
& + \frac{1}{2} [(2a_0 + a_1) \vec{c}_1 - (a_0 - a_1) \vec{c}_2] + a_0 k_1 (\kappa_1 \mp k_2) (\vec{c}_1 \kappa_1 \pm \vec{c}_1 \kappa_2), \\
N^{(2)}_\epsilon = & \quad \vec{c}_1 \vec{c}_2 \left[ \frac{\rho_0 F_0}{2} \omega_0 (a_0 \mp 2a_0) - a_0 k_1 (\kappa_1 \mp k_2) - (\kappa_1 \mp k_2) (a_0 k_1 \mp a_0 k_2) \right] \\
& - \frac{1}{2} [(2a_0 + a_1) \vec{c}_1 - (a_0 - a_1) \vec{c}_2] + a_0 k_1 (\kappa_1 \mp k_2) (\vec{c}_1 \kappa_1 \pm \vec{c}_1 \kappa_2).}
\]