Error Performance Analysis of Space-Time Codes over Rayleigh Fading Channels

Murat Uysal and Costas N. Georghiades

Abstract: **Space-time coding is a bandwidth and power efficient method of communication over fading channels that realizes the benefits of multiple transmit and receive antennas. This novel technique has attracted much attention recently. However, currently the only analytical guide to the performance of space-time codes is an upper bound [1] which could be quite loose in many cases. In this paper, an exact pairwise error probability is derived for space-time codes operating over Rayleigh fading channels. Based on this expression, an analytical estimate for bit error probability is computed, taking into account dominant error events. Simulation results indicate that the estimates are of high accuracy in a broad range of signal-to-noise ratios.**

Index Terms: **Bit-error-rate performance, fading channels, space-time codes, transmitter diversity, pairwise error probability, wireless communication.**

I. INTRODUCTION

Physical limitations on wireless channels present a fundamental technical challenge to reliable communication. Bandwidth limitations, propagation loss, time variance, noise, interference and multipath fading restrict high-data-rate communication over wireless channels. Space-time coding [1]–[3] has been introduced as an effective approach to achieve high data rates. This technique integrates channel coding, modulation and multiple transmit antennas at the base station with optional receive diversity at the mobile station. Performance criteria for space-time codes have been derived in [1] based on the pairwise error probability for both quasi-static flat fading channels, where the path gains are assumed to be constant during a frame, and for rapid fading channels, where the path fading coefficients are assumed to remain constant for one symbol interval.

In this paper, an exact expression for the pairwise error probability (PEP) of space-time codes is derived for rapid fading channels. For this, we make use of the characteristic function technique in [4], [5], which was used previously in the performance analysis of trellis coded modulation (TCM). Our results essentially constitute a generalization of the technique in [4] for the multiple-transmit and multiple-receive antenna case. Therefore, these results also can be applied to TCM schemes employing diversity, as well as space-time coding. The new exact expression is essentially the same as the upper bound derived in [1] with a correction factor, whose value depends on the poles of the Laplace transform of the probability density function of the decision variable. Based on the new PEP, this paper provides analytical results for performance evaluation of space-time coded systems, contrary to the papers in the literature [1]–[3] which only present simulation results. Due to the form of the exact expression, it does not lend itself to the utilization of classical transfer function bounding techniques. Therefore, an estimate of the bit error probability is obtained by taking into account error events up to a finite length, which actually represents the truncation of the infinite series used in calculating the union bound on the bit error probability for high signal-to-noise ratios.

II. SYSTEM MODEL

We consider a wireless communication scenario, where the base station is equipped with M transmit antennas and the mobile unit is equipped with N receive antennas. Data is encoded by a channel encoder and then divided into M parallel streams. Each stream is then passed through an ideal interleaver, i.e., one having infinite interleaving depth. It should be noted that in practice an interleaver with a reasonably large interleaving depth performs as well as an ideal interleaver. The encoded/interleaved data is then pulse shaped and modulated. At the receiver, the received signal is passed to a matched filter having an impulse response with a scaling factor $1/\sqrt{N_0}$ [4], where N_0 is the noise power at the receiver in each receive branch. The output of the matched filter is then sampled at each signaling interval to produce,

$$r_l^n = \sum_{m=1}^{M} \alpha_{l,m}^n x_{l,m} + \eta_l^n, l = 1, 2, \cdots, L, n = 1, 2, \cdots, N, \tag{1}$$

where $x_{l,m}$ is the complex valued modulation symbol transmitted from the m-th transmit antenna in the l-th signaling interval and $\alpha_{l,m}^n$'s are zero mean, complex, Gaussian random variables with a variance of $\sigma_\alpha^2 = E_s/N_0$, which also represents the signal-to-noise ratio (SNR) of the system. The fading variables from each transmit antenna to any receive antenna are assumed to be independent, which is a standard assumption achieved by placing the antennas enough apart. In addition, the interleaver/deinterleaver essentially makes the channel look uncorrelated. The noise samples, η_l^n in (1) are i.i.d. complex Gaussian with unit variance, due to the scaling factor used in the matched

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The authors are with Department of Electrical Engineering, Texas A&M University, College Station, TX, USA, e-mail: uysal@ee.tamu.edu, georghia@ee.tamu.edu.

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filter. The maximum likelihood receiver depends on the minimization of the metric defined as in [1]

$$m(\mathbf{r},\mathbf{x}) = \sum_{l=1}^{L}\sum_{n=1}^{N}\left| r_l^n - \sum_{m=1}^{M} \alpha_{l,m}^n x_{l,m} \right|^2, \tag{2}$$

assuming ideal channel state information is available.

III. DERIVATION OF PAIRWISE ERROR PROBABILITY

The pairwise error probability $P(\mathbf{x}\to\hat{\mathbf{x}})$, which represents the probability of choosing the coded sequence $\hat{\mathbf{x}}$ when indeed $\mathbf{x}$ was transmitted, is given by,

$$P(\mathbf{x}\to\hat{\mathbf{x}}) = \Pr\left[m(\mathbf{r},\hat{\mathbf{x}}) \le m(\mathbf{r},\mathbf{x})\right] = \Pr\left[D \le 0\right], \tag{3}$$

where D is defined as (4) at the bottom of this page.

Inserting (1) into the above expression, D_l^n is given as (5) at the bottom of this page, which is similar to the Eq. (11) in [4] or Eq. (B5) in [6],

$$D_l^n = A_l \left|z_l^n\right|^2 + B_l\left|v_l^n\right|^2 + C_l z_l^n v_l^{n*} + C_l^* z_l^{n*} v_l^n, \tag{6}$$

with the coefficients $A_l = 0, B_l = 1, C_l = -1$, where z_l^n and v_l^n are defined respectively as

$$z_l^n \equiv \eta_l^n, \tag{7}$$

and

$$v_l^n \equiv \sum_{m=1}^{M} \alpha_{l,m}^n \left(\hat{x}_{l,m} - x_{l,m}\right). \tag{8}$$

Recalling that η_l^n and $\alpha_{l,m}^n$ are complex Gaussian variables, $D_l^n$is simply a quadratic form of complex Gaussian variables. Since both spatial and temporal independency holds in our case, the characteristic function of D is

$$\Phi_D(s) = \prod_l \prod_n \phi_l^n(s), \tag{9}$$

where $\phi_l^n(s)$ is the two-sided Laplace transform of the pdf of the random variable D_l^n given by

$$\phi_l^n(s) = \frac{p_{1,l}^n p_{2,l}^n}{\left(s - p_{1,l}^n\right)\left(s - p_{2,l}^n\right)}, \tag{10}$$

with

$$\begin{bmatrix} p_{1,l}^n \\ p_{2,l}^n \end{bmatrix} = w_l^n \mp \sqrt{\left(w_l^n\right)^2 + \frac{1}{4\left(\mu_{zz}\mu_{vv} - \left|\mu_{zv}\right|^2\right)\left(\left|C_l\right|^2 - A_l B_l\right)}}$$

$$w_l^n = \frac{A_l\mu_{zz} + B_l\mu_{vv} + C_l\mu_{zv} + C_l^*\mu_{zv}^*}{4\left(\mu_{zz}\mu_{vv} - \left|\mu_{zv}\right|^2\right)\left(\left|C_l\right|^2 - A_l B_l\right)}.$$

In the above equations, μ_{zz} and μ_{vv} denote the variance of random variable z_l^n and the variance of random variable v_l^n defined in (7) and (8), respectively. μ_{zv} represents the covariance between z_l^n and v_l^n. Using the characteristic function [4], the pairwise error probability is computed as

$$\begin{aligned} P(\mathbf{x}\to\hat{\mathbf{x}}) &= \Pr\left[D \le 0\right] \\ &= -\text{Residue}\left[e^{s\delta}\Phi_D(s)/s\right]\Big|_{\text{Right plane poles, }\delta=0}, \end{aligned} \tag{11}$$

where $\Phi_D(s)$ is the Laplace transform of the pdf of the random variable D. Letting τ denote the time instances where the transmitted and decoded sequences differ, $\Phi_D(s)$ can be expressed as

$$\begin{aligned} \Phi_D(s) = &\left[\prod_{l\in\tau}\left(\frac{E_s}{4N_0}\sum_{m=1}^{M}\left|\hat{x}_{l,m} - x_{l,m}\right|^2\right)\right]^{-N} \\ &\times \left[\prod_{l\in\tau}\frac{-1}{16\left(s - p_{1,l}\right)\left(s - p_{2,l}\right)}\right]^N, \end{aligned} \tag{12}$$

with

$$\begin{bmatrix} p_{1,l} \\ p_{2,l} \end{bmatrix} = \frac{1}{4} \mp \sqrt{\frac{1}{16} + \frac{1}{4\frac{E_s}{N_o}\sum_{m=1}^{M}\left|\hat{x}_{l,m} - x_{l,m}\right|^2}}.$$

It is worth noting that the first product term in (12) is the upper bound in [1]. In other words, the exact pairwise error event probability is the upper bound derived in [1] modified by a correction factor given by the second product term whose value depends on the poles of $\Phi_D(s)$.

It is obvious that the pairwise error probability is not the main issue in performance evaluation of a digital communication system. In the case that the pairwise error probability can be expressed in a product form, it is possible to find an upper bound

$$D = \sum_{l=1}^{L}\sum_{n=1}^{N} D_l^n = \sum_{l=1}^{L}\sum_{n=1}^{N}\left\{\left|r_l^n - \sum_{m=1}^{M}\alpha_{l,m}^n \hat{x}_{l,m}\right|^2 - \left|r_l^n - \sum_{m=1}^{M}\alpha_{l,m}^n x_{l,m}\right|^2\right\}. \tag{4}$$

$$D_l^n = \left|\sum_{m=1}^{M}\alpha_{l,m}^n\left(\hat{x}_{l,m} - x_{l,m}\right)\right|^2 - \sum_{m=1}^{M}\alpha_{l,m}^n\left(\hat{x}_{l,m} - x_{l,m}\right)\left(\eta_l^n\right)^* - \sum_{m=1}^{M}\left(\alpha_{l,m}^n\right)^*\left(\hat{x}_{l,m} - x_{l,m}\right)^*\eta_l^n, \tag{5}$$

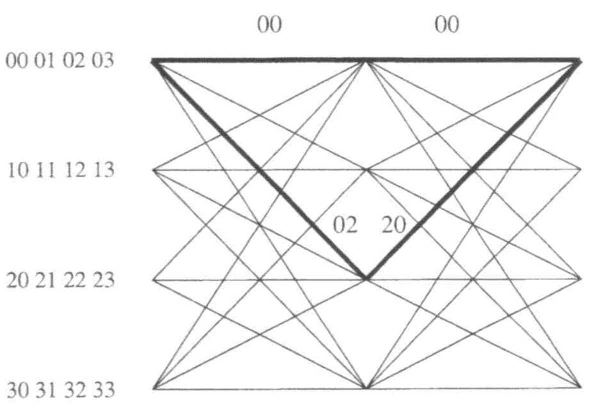

Fig. 1. Space-time code (4-PSK, 4-state).

on the bit error probability by using the transfer function approach. The transfer function method [6] is a classical technique which makes use of the code's state diagram to obtain error rate performance for trellis based codes (i.e., convolutional and TCM). Here, however, resulting pairwise error probability (i.e., Eq. (11)) is given in terms of a residue computation and, therefore, does not lend itself to the utilization of classical transfer function upper bounding techniques. Thus, instead of using transfer function approach which takes into account error events of all lengths, an estimation of bit error probability can be obtained through accounting for error event paths of lengths up to a pre-determined specific value as follows [4]:

$$P_b \approx \frac{1}{k} \sum_{\hat{\mathbf{x}} \neq \mathbf{x}} q\left(\mathbf{x} \rightarrow \hat{\mathbf{x}}\right) P\left(\mathbf{x} \rightarrow \hat{\mathbf{x}}\right), \tag{13}$$

where k is the number of input bits per trellis transition and $q\left(\mathbf{x} \rightarrow \hat{\mathbf{x}}\right)$ is the number of bit errors associated with each error event. If the maximum length of error events taken into account is chosen as H, it is sufficient to consider the error events of length up to H; this represents a truncation of the infinite series used in calculating the union bound on the bit error probability for high SNR values. The choice of H is critical in the sense that most of the dominant error events for the SNR range of interest should be taken into account by a proper choice while preventing excessive computational complexity, which grows roughly in an exponential manner with H. We should point out that, here, the analytical estimate is obtained under the assumption that the transmitted code sequence is the all zero codeword, which allows us to compute the approximation considering only the error events that diverge from the zero path. Actually, the bit error probability depends on the transmitted code sequence. However, it is reported in [7] that computation based on a fixed reference sequence provides satisfactory results. It should also be emphasized that (13) does not provide an upper bound on the bit error probability. Therefore, the actual results or the simulation results can be lower or higher than the approximation [5].

IV. EXAMPLES

In this section, two different space-time codes proposed in [1] are considered as examples. The first space-time code under consideration is illustrated in Fig. 1. We assign the input bits $\{00, 01, 10, 11\}$ to trellis branches emanating from a specific node. The labeling of the branches follow [1]. If the label of a specific branch is $s_{l,1}, s_{l,2}, \cdots, s_{l,M}$, then transmit antenna m is used to send the constellation symbol $s_{l,m}$, $m = 1, 2, \cdots, M$, and all these transmissions are simultaneous.

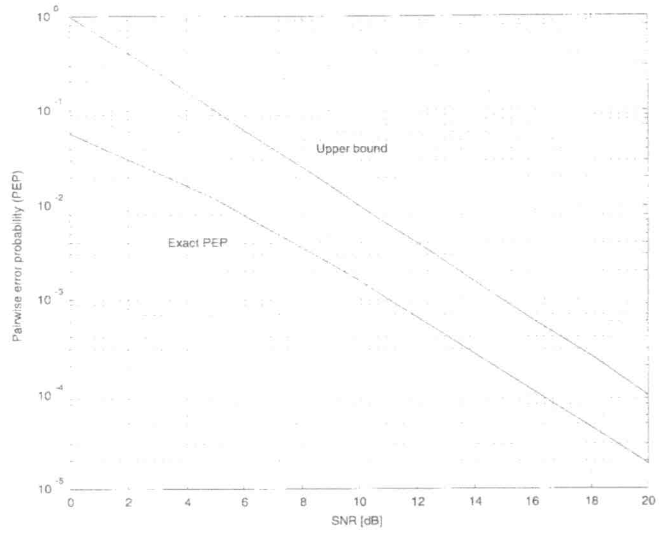

Fig. 2. Exact pairwise error probability for error event of length two.

As an example to demonstrate the difference between the exact PEP and the upper bound on PEP provided in [1], we consider a specific error event for PEP computation. Recalling that the length of an error event is the number of incorrect nodes in the path which diverges from the all-zero sequence, it is seen that the length of the shortest error event for this code is 2. An example of a shortest error event is illustrated by shading in Fig. 1. This corresponds to the case when the transmitted sequence is all zero-phase codeword and the erroneous sequence is $\{1, -1, -1, 1\}$, noting that $\exp(jk\pi/2)$ is the complex value for the k-th signal point in the signal constellation. The Chernoff upper bound for this event is found to be

$$P\left(\mathbf{x} \rightarrow \hat{\mathbf{x}}\right) \left(E_s/N_o\right)^{-2}. \tag{14}$$

The exact pairwise error probability for this specific error event is also evaluated by using (11), (12) and both are plotted in Fig. 2 for comparison. It is observed from Fig. 2 that the upper bound is quite loose when compared to the exact expression.

We also evaluate the bit error probability performance for this code, both based on the upper bound on pairwise error probability derived in [1] which has a product form and allows the use of the transfer function bounding technique, and based on the exact pairwise error probability given by (11), (12) used in conjunction with (13). The upper bound on the bit-error probability is given by the union bound as in [8]

$$P_b \leq \frac{1}{k} \frac{\partial T\left(D, I\right)}{\partial I}\bigg|_{I=1}, \tag{15}$$

where $T(D, I)$ is the transfer function of the error state diagram

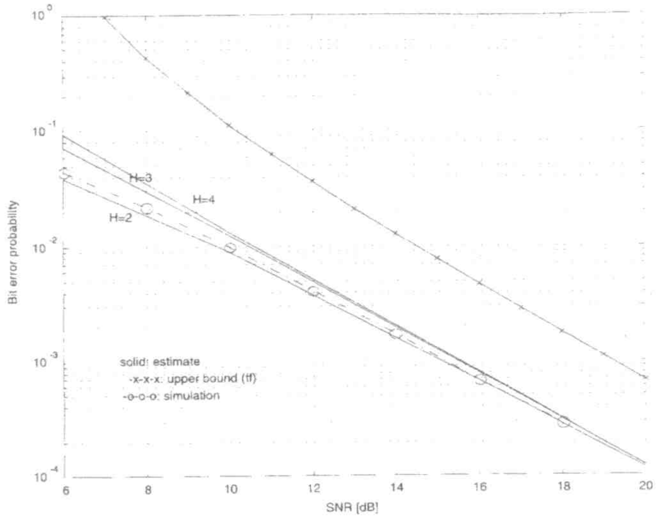

Fig. 3. Performance of four-state code with two transmit and one receive antennas.

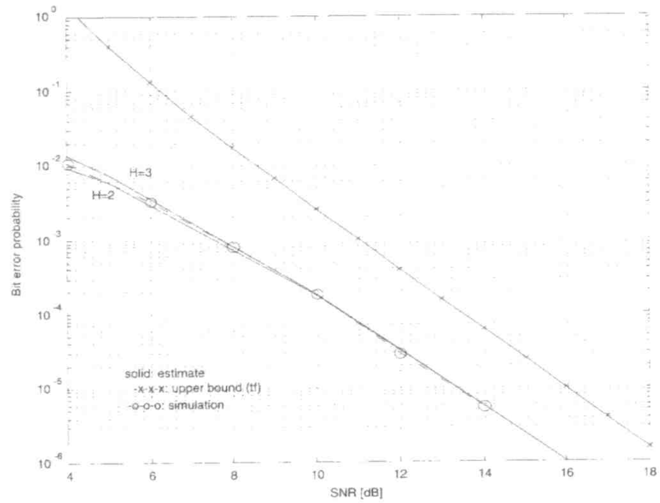

Fig. 4. Performance of four-state code with two transmit and two receive antennas.

and D is given by [1]

$$D = \left(\frac{E_s}{4N_0} \cdot \sum_{m=1}^{M} |\hat{x}_{l,m} - x_{l,m}|^2 \right)^{-N}. \tag{16}$$

The upper bound on bit error probability is illustrated in Fig. 3 for $N = 1$ and in Fig. 4 for $N = 2$. In both cases, two transmit antennas are used (i.e., $M = 2$). In order to obtain an estimate of the bit error probability based on (13) in conjunction with the exact pairwise error probability derived, first all error events that diverge from the all zero codeword with length less than or equal to 4 (i.e., in this example H is set to 4) are determined. The exact probability of each event is calculated by using (11), then these values are inserted into (13) to obtain the estimate. The analytical estimates of bit error probability taking into account the error events up to length $H = 2, 3,$ and 4 are plotted in Figs. 3 and 4 for the space-time code with two transmit antennas and with one or two receive antennas. The results obtained through the transfer function technique and simulations are presented as well. From Figs. 3 and 4, it is clearly seen that the simulation results agree very well with the analytical estimates while results provided through the transfer function technique provide a loose upper bound. The inclusion of estimates computed with different H parameters also allows us to determine which value of this parameter is sufficient to obtain good estimates. In this example, even $H = 2$ provides satisfactory results indicating that most of the dominant error events at the SNR of interest are included. It should be also noted that all the estimates, each computed based on different H values, overlap for large SNR values.

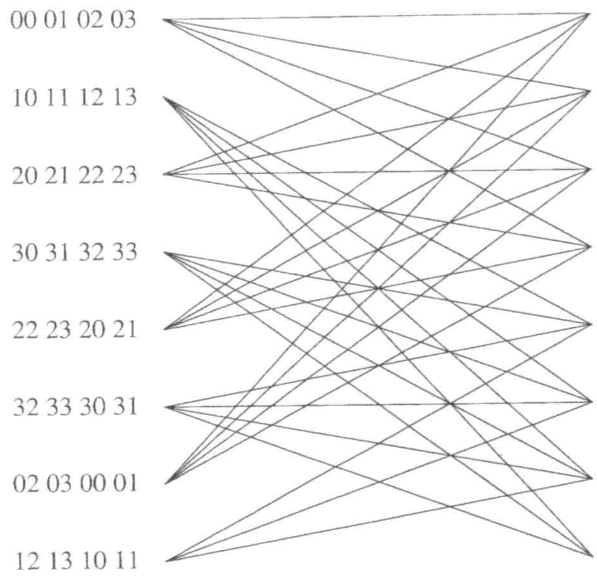

Fig. 5. Space-time code (4-PSK, 8-state).

As a second example, consider the 8-state space-time code [1] illustrated in Fig. 5. The estimates based on exact pairwise error probability are plotted in Figs. 6 and 7 for the cases with one or two receive antennas respectively. Corresponding simulation results are demonstrated as well. The analytical and simulation results are in good agreement as in the previous example. The figures also reveal that for this example H should be chosen to be at least 3 in order to obtain a satisfactory estimate.

V. CONCLUSION

In this paper, we provide analytical tools for the evaluation of space-time codes operating over fading channels. First, an exact expression of pairwise error probability is derived for space-time codes over rapid fading channels where the path gains are assumed to be constant over one symbol interval. This exact expression is essentially the same as the upper bound derived in [1] with a correction factor whose value is dependent on the poles of the Laplace transform of the density function of the decision variable. Based on this expression, an analytical estimate for bit error probability is computed, taking into account a small number of error events. Simulation results demonstrate that the estimates are of high accuracy in the SNR range of interest. The extension of this approach for a multi-user environment can be found in [9].

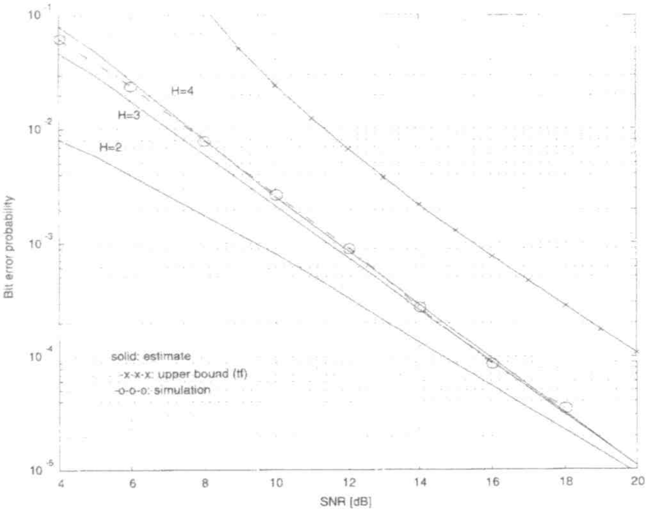

Fig. 6. Performance of eight-state code with two transmit and one receive antennas.

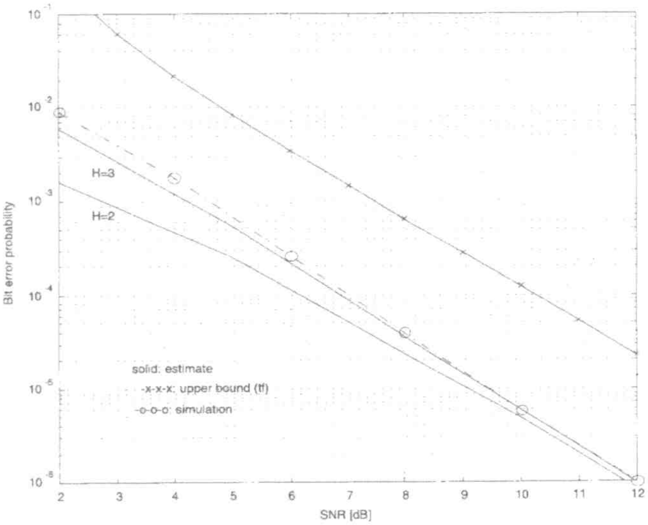

Fig. 7. Performance of eight-state code with two transmit and two receive antennas.

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Murat Uysal was born in Istanbul, Turkey in 1973. He received his B.Sc. and M.Sc. degrees in Electronics and Communication Engineering from Istanbul Technical University, Turkey, in 1995 and 1998, respectively. He worked as a research assistant in the Department of Electronics and Communication Engineering at Istanbul Technical University from 1995 to 1998. Since August 1998, he has been with the Department of Electrical Engineering at Texas A&M University, College Station, Texas, pursuing Ph.D. degree in electrical engineering. His current interests are on the design and performance analysis of space-time codes and receiver design for space-time coded communication systems.

Costas N. Georghiades was born in Cyprus in 1955. He received the B.E. degree with distinction from the American University of Beirut in June 1980, and the M.S. and D.Sc. degrees from Washington University in May 1983 and May 1985, respectively, all in Electrical Engineering. Since September 1985 he has been with the Electrical Engineering department at Texas A&M University, where he is a Professor and holder of the J.W. Runyon, Jr. Endowed Professorship and Director of the Telecommunications and Signal Processing Group. His general interests are in the application of information, communication, and estimation theories to the study of communication systems.

Dr. Georghiades is a Fellow of the IEEE, a member of Sigma Xi and Eta Kappa Nu, and a registered Professional Engineer in Texas. He served as an associate editor for the IEEE Transactions on Communications, as Publications Editor and Associate Editor for Communications for the IEEE Transactions on Information Theory, as Guest-Editor for the IEEE Journal on Selected Areas in Communications and as an Associate Editor for the IEEE Communications Letters. He has been involved in organizing a number of conferences, including as Technical Program Chair for the 1999 IEEE Vehicular Technology Conference. He currently serves as program chair for the 2001 Communication Theory Workshop, and as Chair for the Communication Theory Symposium for GLOBECOM 2001.