On the Error Performance Analysis of Space–Time Trellis Codes

Murat Uysal, *Member, IEEE,* and Costas N. Georghiades, *Fellow, IEEE*

***Abstract*—We present analytical performance results for space–time trellis codes over spatially correlated Rayleigh fading channels. Bit-error-probability estimates are obtained based on the derivation of an exact pairwise error probability expression using a residue technique combined with a characteristic function approach. We investigate both quasi-static and interleaved channels and demonstrate how the spatial fading correlation affects the performance of space–time codes over these two different channel models. Simulation results are also included to confirm the accuracy of analytical estimates.**

***Index Terms*—Bit-error rate, fading channels, pairwise error probability (PEP), space–time trellis coding.**

I. Introduction

Space–time trellis coding [1] has been proposed as an effective approach to support high data rate transmission over fading channels. Since its introduction, a significant amount of work has been published on the design, decoding, and applications of this new family of codes. Particular attention has been paid on the analytical performance evaluation of space–time coding [3]–[6]. In the original work [1], performance criteria for space–time codes were derived based on an upper bound on the pairwise error probability (PEP) for both quasi-static flat fading channels, where the path gains are assumed constant during a frame, and for symbol-by-symbol ideally interleaved channels, where the path gains are assumed constant over a symbol interval. For the latter case, the authors derived an exact PEP expression [6] by making use of the residue technique combined with the characteristic function approach in [7] and [8], which was used previously in the performance analysis of trellis-coded modulation. Those results essentially constitute a generalization of the technique in [8] for the multiple-transmit and multiple- receive antenna case. Extensions to investigate the effect of spatial fading correlation and multiple access interference on performance were reported in [9] and [10]. In this paper, using the matrix version of quadratic form of complex Gaussian random variables [11], [12], we extend our results to accommodate temporally correlated fading channels with our previous results in [6] and [9] being special cases. Based on the derived PEP, this paper provides analytical performance results for space–time coded systems over spatially correlated/uncorrelated quasi-static channels as well as perfectly interleaved channels. Straightforward manipulations of the exact expressions for the PEP derived in this work do validate earlier results—like the determinant [1] and equal eigenvalues [5] criteria, which had been initially derived from bounds on the PEP—while factoring in, analytically, the effect of correlation between the transmit antennas.

Manuscript received December 10, 2001; revised May 21, 2002, March 17, 2003, April 29, 2003, and May 7, 2003; accepted May 7, 2003. The editor coordinating the review of this paper and approving it for publication is P. F. Driessen. This paper was presented in part at IEEE Wireless Communications and Networking Conference (WCNC), Orlando, FL, March 2002.

M. Uysal is with the Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, ON N2G-3L1, Canada (e-mail: muysal@ece.uwaterloo.ca).

C. N. Georghiades is with the Department of Electrical Engineering, Texas A&M University, College Station, TX 77843 USA (e-mail: georghia@ee.tamu.edu).

Digital Object Identifier 10.1109/TWC.2004.830824

II. System Model

Let M and N be the number of transmit and receive antennas, respectively. The binary data stream is first modulated and mapped to a sequence of complex modulation symbols. The modulated sequence is then fed to the space–time encoder. At the receiver, the received signal is passed to a matched filter having an impulse response with a scaling factor $1/\sqrt{N_0}$ [8], where N_0 is the noise power at the receiver in each receive branch. The output of the matched filter is then sampled at each signaling interval to produce

$$r_n^l = \sum_{m=1}^{M} \alpha_{m,n}^l x_m^l + \eta_n^l,\ l = 1, 2, \dots, L, \qquad n = 1, 2, \dots, N \tag{1}$$

where x_m^l is the complex valued modulation symbol transmitted from the mth transmit antenna in the lth signaling interval and $\alpha_{m,n}^l$ is the fading coefficient modeling the channel from the mth transmit to the nth receive antenna. The noise samples η_n^l in (1) are independent and identically distributed complex Gaussian random variables and independent of the fading coefficients. Their variance is equal to unity due to the scaling factor used in the matched filter [8]. The fading coefficients are modeled as zero-mean complex Gaussian random variables with a variance of $\sigma_\alpha^2 = \overline{E}_s/N_0 = (1/M)(E_s/N_0)$. Here E_s/N_0 coincides with the average signal-to-noise ratio (SNR) per receive antenna. Assuming coherent detection with perfect channel knowledge, the maximum likelihood receiver depends on the minimization of the metric [1]

$$\mu(\boldsymbol{r}, \boldsymbol{X}) = \sum_{l=1}^{L} \sum_{n=1}^{N} \left| r_n^l - \sum_{m=1}^{M} \alpha_{m,n}^l x_m^l \right|^2 \tag{2}$$

where $\boldsymbol{r}$ is the vector consisting of the received signals collected from N receive antennas for a frame of L symbols. The Viterbi algorithm is then used to compute the path with the lowest accumulated metric. Denoting $\Gamma_{m,n}$ as the vector space of m-by-n complex matrices and defining[1]

$$\begin{aligned}\boldsymbol{\alpha}_n^l &= \left(\alpha_{1,n}^l, \alpha_{2,n}^l, \ldots, \alpha_{M,n}^l\right)^T \in \Gamma_{M,1}\\ \boldsymbol{\alpha}_n &= \left(\boldsymbol{\alpha}_n^{1\,T}, \boldsymbol{\alpha}_n^{2\,T}, \ldots, \boldsymbol{\alpha}_n^{L\,T}\right)^T \in \Gamma_{ML,1}\\ \boldsymbol{r}_n &= \left(r_n^1, r_n^2, \ldots, r_n^L\right)^T \in \Gamma_{L,1}\\ \boldsymbol{\eta}_n &= \left(\eta_n^1, \eta_n^2, \ldots, \eta_n^L\right)^T \in \Gamma_{L,1}\\ \boldsymbol{x}^l &= \left(x_1^l, x_2^l, \ldots, x_M^l\right) \in \Gamma_{1,M}\end{aligned}$$

and

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}^1 & \mathbf{0}_{1\times M} & \ldots & \mathbf{0}_{1\times M} \\ \mathbf{0}_{1\times M} & \boldsymbol{x}^2 & \ldots & \mathbf{0}_{1\times M} \\ \cdot & \cdot & \cdot & \cdot \\ \mathbf{0}_{1\times M} & \mathbf{0}_{1\times M} & \ldots & \boldsymbol{x}^L \end{bmatrix} \in \Gamma_{L,ML}$$

we can write the received signal at the nth receive antenna in matrix notation as

$$\boldsymbol{r}_n = \boldsymbol{X}\boldsymbol{\alpha}_n + \boldsymbol{\eta}_n, \qquad n = 1, 2, \ldots, N. \tag{3}$$

In this case, the metric in (2) can be rewritten as

$$\mu(\boldsymbol{r}, \boldsymbol{X}) = \sum_{n=1}^{N} \|\boldsymbol{r}_n - \boldsymbol{X}\boldsymbol{\alpha}_n\|^2. \tag{4}$$

III. Derivation of PEP

The PEP $P(\boldsymbol{X} \to \hat{\boldsymbol{X}})$, which represents the probability of choosing the coded sequence $\hat{\boldsymbol{X}}$ when indeed $\boldsymbol{X}$ was transmitted, is given by

$$P\left(\boldsymbol{X} \to \hat{\boldsymbol{X}}\right) = \Pr\left[\mu\left(\boldsymbol{r}, \hat{\boldsymbol{X}}\right) \le \mu(\boldsymbol{r}, \boldsymbol{X})\right] = \Pr[D \le 0] \tag{5}$$

where D is defined as

$$D = \sum_{n=1}^{N} \left(\left\|\boldsymbol{r}_n - \hat{\boldsymbol{X}}\boldsymbol{\alpha}_n\right\|^2 - \|\boldsymbol{r}_n - \boldsymbol{X}\boldsymbol{\alpha}_n\|^2 \right). \tag{6}$$

Inserting (3) into (6) and expanding the resulting terms, D becomes

$$D = \sum_{n=1}^{N} \left(\boldsymbol{\alpha}_n^H \left(\hat{\boldsymbol{X}} - \boldsymbol{X}\right)^H \left(\hat{\boldsymbol{X}} - \boldsymbol{X}\right) \boldsymbol{\alpha}_n - \boldsymbol{\alpha}_n^H \left(\hat{\boldsymbol{X}} - \boldsymbol{X}\right)^H \boldsymbol{\eta}_n - \boldsymbol{\eta}_n^H \left(\hat{\boldsymbol{X}} - \boldsymbol{X}\right) \boldsymbol{\alpha}_n \right) \tag{7}$$

which is the matrix version of [6, eq. (5)]. The PEP is computed as [7], [8]

$$P\left(\boldsymbol{X} \to \hat{\boldsymbol{X}}\right) = \Pr[D \le 0] = -\text{Residue}\left[\frac{\Phi_D(s)}{s}\right]\Bigg|_{\text{Right plane poles}} \tag{8}$$

[1]Throughout this paper, we use $(\cdot)^T$ and $(\cdot)^H$ to represent transpose and conjugate transpose operations, respectively. $\mathbf{0}_{m\times n} \in \Gamma_{m,n}$ and $\boldsymbol{I}_{m\times n} \in \Gamma_{m,n}$ are defined as zero and identity matrices with appropriate dimensions.

where $\Phi_D(s)$ is the Laplace transform of the pdf of random variable D, which is given in (7) as a summation over the quadratic form of complex Gaussian variables. If we define the following matrices:

$$\begin{aligned}\boldsymbol{y}_n &= \begin{bmatrix} \boldsymbol{\eta}_n^H & \left(\left(\hat{\boldsymbol{X}} - \boldsymbol{X}\right)\boldsymbol{\alpha}_n\right)^H \end{bmatrix}^H \in \Gamma_{2L,1}\\ \boldsymbol{K} &= \begin{bmatrix} \mathbf{0}_{LxL} & -\boldsymbol{I}_{LxL} \\ -\boldsymbol{I}_{LxL} & \boldsymbol{I}_{LxL} \end{bmatrix} \in \Gamma_{2L,2L}\end{aligned}$$

an alternative representation for D can be obtained as

$$D = \sum_{n=1}^{N} \boldsymbol{y}_n^H \boldsymbol{K} \boldsymbol{y}_n. \tag{9}$$

The characteristic function of D is then given as [11], [12]

$$\Phi_D(s) = \prod_{n=1}^{N} \frac{1}{\det\left(\boldsymbol{I}_{2L\times 2L} + s\boldsymbol{C}_{\boldsymbol{y}_n}\boldsymbol{K}\right)} \tag{10}$$

where $\boldsymbol{C}_{\boldsymbol{y}_n}$ is the covariance matrix of $\boldsymbol{y}_n$. Considering the independence of $\left(\hat{\boldsymbol{X}} - \boldsymbol{X}\right)\boldsymbol{\alpha}_n$ and $\boldsymbol{\eta}_n$, which constitute the quadratic form, $\boldsymbol{C}_{\boldsymbol{y}_n}$ is found as

$$\boldsymbol{C}_{\boldsymbol{y}_n} = \begin{bmatrix} \boldsymbol{I}_{LxL} & \mathbf{0}_{LxL} \\ \mathbf{0}_{LxL} & \boldsymbol{C}_{d_n} \end{bmatrix} \tag{11}$$

where $\boldsymbol{C}_{d_n}$ is the covariance matrix of $\left(\hat{\boldsymbol{X}} - \boldsymbol{X}\right)\boldsymbol{\alpha}_n$. Furthermore, defining $\boldsymbol{C}_{\alpha_n}$ as the covariance matrix of $\boldsymbol{\alpha}_n$, we obtain $\boldsymbol{C}_{d_n} = \left(\hat{\boldsymbol{X}} - \boldsymbol{X}\right)\boldsymbol{C}_{\alpha_n}\left(\hat{\boldsymbol{X}} - \boldsymbol{X}\right)^H$. The (l, k) term of $\boldsymbol{C}_{d_n}$, $l = 1, 2, \ldots, L$, $k = 1, 2, \ldots, L$, is given as

$$\boldsymbol{C}_{d_n}(l,k) = E\left[\sum_{m=1}^{M}\sum_{q=1}^{M} \alpha_{m,n}^l \alpha_{q,n}^{k\;*} \left(\hat{x}_m^l - x_m^l\right)\left(\hat{x}_q^k - x_q^k\right)^*\right] \tag{12}$$

where $E[\cdot]$ is the expectation operation.

IV. Case Studies

In this section, we investigate two different cases, namely, the symbol-by-symbol interleaved channel and the quasi-static channel.

A. Spatially Correlated Symbol-by-Symbol Interleaved Channel

We assume the fading coefficients are constant over one symbol interval and change from one symbol to another independently. Such an assumption can be justified by the use of perfect interleaving. This is the case referred to as "fast fading" in [1] and also investigated in [6] and [9]. We assume spatial correlation between the transmit antennas while assuming independency between receive antennas. Since symbol-by-symbol interleaving guarantees temporal independence, $\boldsymbol{C}_{d_n}$ is a diagonal matrix, i.e., $\boldsymbol{C}_{d_n} = diag(\beta_1, \beta_2, \ldots, \beta_L)$ with

$$\beta_l = \frac{\overline{E}_s}{N_0}\left(\sum_{m=1}^{M}\left|\hat{x}_m^l - x_m^l\right|^2 + \sum_{\substack{m=1\\ m\neq q}}^{M}\sum_{q=1}^{M} \rho_{m,q}^l (\hat{x}_m^l - x_m^l)(\hat{x}_q^l - x_q^l)^*\right) \tag{13}$$

where $\rho_{m,q}^{l}$ is the normalized spatial correlation factor between fading coefficients $\alpha_{m,n}^{l}$ and $\alpha_{q,n}^{l}$. Therefore, the inner term in the determinant of (10) reduces to a form

$$\boldsymbol{I}_{2L\times 2L}+s\boldsymbol{C}_{\boldsymbol{y}_n}\boldsymbol{K} =\begin{bmatrix} \boldsymbol{I}_{L\times L} & -s\boldsymbol{I}_{L\times L} \\ -s\cdot diag(\beta_1,\dots\beta_L) & (\boldsymbol{I}_{LxL}+s\cdot diag(\beta_1,\dots\beta_L)) \end{bmatrix}.$$

In this case, it can be easily shown that (e.g., by induction)

$$\det\left(\boldsymbol{I}_{2L\times 2L}+s\boldsymbol{C}_{\boldsymbol{y}_n}\boldsymbol{K}\right)=\prod_{l=1}^{L}\left(1+s\beta_l-s^2\beta_l\right). \tag{14}$$

Therefore, the characteristic function can be computed in a product form over time instances

$$\Phi_D(s)=\prod_{l=1}^{L}\left(\frac{1}{1+s\beta_l-s^2\beta_l}\right)^N=\prod_{l=1}^{L}\left(\frac{1}{\beta_l}\frac{-1}{s^2-s-\frac{1}{\beta_l}}\right)^N. \tag{15}$$

Expanding the denominator, we obtain

$$\Phi_D(s)=\left[\prod_{l=1}^{|\varpi|}\frac{\beta_l}{4}\right]^{-N}\left[\prod_{l=1}^{|\varpi|}\frac{-1}{4\left(s-p_{1,l}\right)\left(s-p_{2,l}\right)}\right]^N \tag{16}$$

with

$$\begin{bmatrix} p_{1,l} \\ p_{2,l} \end{bmatrix}=\frac{1}{2}\mp\sqrt{\frac{1}{4}+\frac{1}{\beta_l}}$$

where ϖ is the set of time instances where the transmitted and decoded sequences differ and $|\varpi|$ stands for the number of elements in ϖ. Equation (16) was previously obtained in [9] starting from the scalar version of the quadratic form given in [7]. It is worth noting that by setting $\rho_{m,q}^{l}=0$ (i.e., no spatial correlation) in (16), we obtain the spatially independent case investigated in [6]. As emphasized in [6], the first product term in (16) is the upper bound in [1]. In other words, the exact PEP is the upper bound derived in [1] modified by a correction factor given by the second product term whose value depends on the poles of $\Phi_D(s)$. The form of (16) also reveals maximum diversity order (i.e., slope of the performance curve) which can be achieved for a space–time trellis coded system over an interleaved channel. At large SNRs, the error performance curve is dominated by the shortest error events. Denoting $|\varpi'|$ as the length of the shortest error event, the error performance curve will vary with $\left(\overline{E}_s/N_0\right)^{-|\varpi'|N}$ asymptotically. Therefore, the product of the number of receive antennas and the length of the shortest error event of the underlying trellis code (i.e., $|\varpi'|N$) determine the diversity order. It is also seen from (16) that the use of multiple transmit antennas over the interleaved channel only manifests itself as an increase in the product distance, which affects the coding gain (i.e., horizontal shift in the performance curve), but not the diversity order. It is also observed that spatial correlation between the transmit antennas only results in a change in coding gain, but does not affect the diversity order.

B. Spatially Correlated Quasi-Static Channel

Now, we assume that the channel fading coefficients remain constant over one frame (i.e., a number of symbols) and change from one frame to another independently. The (l,k) term of $\boldsymbol{C}_{d_n}$ is given in (17), shown at the bottom of the page, where $\rho_{m,q}$ is the normalized spatial correlation factor between fading coefficients $\alpha_{m,n}$ and $\alpha_{q,n}$. The time index in fading coefficients are dropped due to the quasi-static assumption. $\boldsymbol{C}_{d_n}$ is no more diagonal due to the introduced temporal correlation between fading gains and (15) does not hold any more. Using determinant properties [14, p. 478] and exploiting the special structures of $\boldsymbol{C}_{y_n}$ and $\boldsymbol{K}$, (10) can be rewritten as

$$\Phi_D(s)=\left(\frac{1}{\det\left(\boldsymbol{I}_{2L\times 2L}+s\boldsymbol{C}_{\boldsymbol{y}_n}\boldsymbol{K}\right)}\right)^N =\left(\frac{1}{\det\left(\boldsymbol{I}_{2L\times 2L}+(s-s^2)\,\boldsymbol{C}_{d_n}\right)}\right)^N. \tag{18}$$

Furthermore, using properties of eigenvalues [14, pp. 36–42], we obtain

$$\Phi_D(s)=\prod_{l=1}^{\gamma}\left(\frac{1}{1+(s-s^2)\lambda_l}\right)^N=\prod_{l=1}^{\gamma}\left(\frac{1}{\lambda_l}\frac{-1}{s^2-s-\frac{1}{\lambda_l}}\right)^N. \tag{19}$$

Here, λ_l are the nonzero eigenvalues of $\boldsymbol{C}_{d_n}=\left(\hat{\boldsymbol{X}}-\boldsymbol{X}\right)\boldsymbol{C}_{\alpha_n}\left(\hat{\boldsymbol{X}}-\boldsymbol{X}\right)^H$. A similar result is reported in [4] for the spatially uncorrelated case. For this case, we can write $\boldsymbol{C}_{d_n}=\left(\overline{E}_s/N_0\right)(\hat{\boldsymbol{\Omega}}-\boldsymbol{\Omega})(\hat{\boldsymbol{\Omega}}-\boldsymbol{\Omega})^H$, where $\boldsymbol{\Omega}$ is defined as $\boldsymbol{\Omega}=\left[\boldsymbol{x}^1,\boldsymbol{x}^2,\dots,\boldsymbol{x}^L\right]^T\in\Gamma_{L,M}$.

Equation (19) reveals that the achievable maximum diversity order is determined by the value of γN, namely the product of the number of nonzero eigenvalues of $\boldsymbol{C}_{d_n}$ and the number of receive antennas. It can be easily seen that the rank of $\boldsymbol{C}_{\alpha_n}$ (due to its special form for quasi-static case), therefore the rank of $\boldsymbol{C}_{d_n}$, is limited by M, (i.e., maximum M out of total L eigenvalues turn out to be nonzero). This also agrees with the original criteria derived in [1] by using a Chernoff bound. It is also worth emphasizing the similarity of (19) to (15). For interleaved channel $\boldsymbol{C}_{\alpha_n}$ further reduces to a diagonal form, which also makes $\boldsymbol{C}_{d_n}$ diagonal with diagonal elements equal to β_l given in (13). Since the eigenvalues of a diagonal matrix are equal to its diagonal elements, we can simply obtain (15) replacing λ_l with β_l in (19).

$$\boldsymbol{C}_{d_n}(l,k)=\begin{cases} \frac{\overline{E}_s}{N_0}\left(\sum\limits_{m=1}^{M}\left|\hat{x}_m^l-x_m^l\right|^2+\sum\limits_{\substack{m=1\\m\neq q}}^{M}\sum\limits_{q=1}^{M}\rho_{m,q}\left(\hat{x}_m^l-x_m^l\right)\left(\hat{x}_q^l-x_q^l\right)^*\right), & k=l \\ \frac{\overline{E}_s}{N_0}\left(\sum\limits_{m=1}^{M}\left(\hat{x}_m^l-x_m^l\right)\left(\hat{x}_m^k-x_m^k\right)^*+\sum\limits_{\substack{m=1\\m\neq q}}^{M}\sum\limits_{q=1}^{M}\rho_{m,q}\left(\hat{x}_m^l-x_m^l\right)\left(\hat{x}_q^k-x_q^k\right)^*\right), & k\neq l \end{cases} \tag{17}$$

V. Bit-Error-Probability Performance

When PEP can be expressed in product form, it is possible to find an upper bound on the bit-error probability by using the transfer function approach [7]. Here, however, PEP [i.e., (8)] is given in terms of a residue computation and, therefore, does not lend itself to the utilization of classical transfer function upper bounding techniques. Thus, instead of using the transfer function approach which implicitly takes into account error events of all lengths, an estimation of bit-error probability can be obtained through accounting for error event paths of lengths up to a predetermined specific value as follows [8]

$$P_b \approx \frac{1}{k} \sum_{\boldsymbol{X} \neq \hat{\boldsymbol{X}}} q\left(\boldsymbol{X} \to \hat{\boldsymbol{X}}\right) P\left(\boldsymbol{X} \to \hat{\boldsymbol{X}}\right) \tag{20}$$

where k is the number of input bits per trellis transition and $q\left(\boldsymbol{X} \to \hat{\boldsymbol{X}}\right)$ is the number of bit errors associated with each error event. Equation (20) essentially represents a truncation of the infinite series used in calculating the union bound on the bit-error probability. The maximum length of error events taken into account, say v, can be chosen considering the properties of the code under investigation. The performance is mainly dominated by the error events that diverge from the correct path with the smallest number of symbols at nonzero Euclidean distances. This can be defined as the effective code length (ECL) using a similar terminology to [13]. Therefore, at least the error events with ECL should be taken into account to obtain a sufficient estimate. Besides ECL, the distribution of error events may also affect the overall bit-error-probability performance. For the spatially correlated case, the choice of v should be made in such a way that at least nonorthogonal error events of the shortest length are taken into account. Otherwise, it is not possible to observe the correlation effect through just considering orthogonal error events. It should be mentioned that the shortest length of nonorthogonal events may be higher than the ECL of the code. As a final note, it should be emphasized that (20) does not provide an upper bound on the bit-error probability. The actual results or the simulation results can be lower or higher than the approximation [15]. Although our discussion here focuses on the bit-error performance, one can use PEP expressions in evaluating similar estimates for the frame-error probability by a proper truncation of [16, eq. (14)].

VI. Example: Four-State QPSK Space–Time Code

First, we consider the PEP computation for the shortest error event of the four-state quadrature-phase-shift keying (QPSK) space–time code proposed in [1] as an example. The trellis diagram and the considered error event can be found in [6, Fig. 1]. The exact PEP for this specific error event is evaluated by using (8) in conjunction with (16) for interleaved channel and with (19) for quasi-static case. Computing the residues required in (8) and after some mathematical manipulation, we obtain

$$P\left(\boldsymbol{X} \to \hat{\boldsymbol{X}}\right) = \frac{1}{4}\left(2 + \sqrt{\frac{\frac{\overline{E}_s}{N_0}}{\frac{\overline{E}_s}{N_0}+1}}\right)\left(1 - \sqrt{\frac{\frac{\overline{E}_s}{N_0}}{\frac{\overline{E}_s}{N_0}+1}}\right)^2. \tag{21}$$

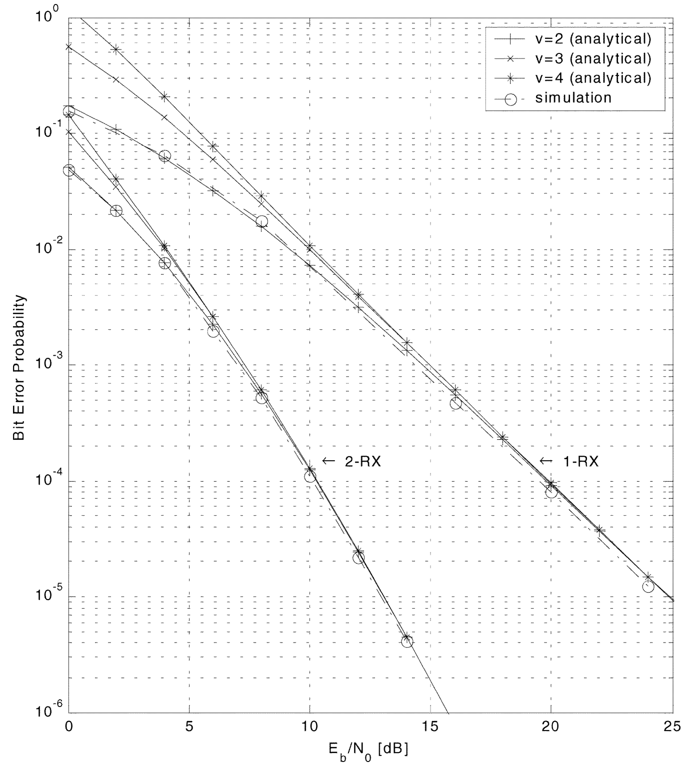

Fig. 1. Performance of the four-state space–time trellis code (STTC) over interleaved Rayleigh fading channels.

It is interesting to note that (21) corresponds to [7, eq. (14.4)–(15)] with $L = 2$, which is an error probability expression obtained for receive diversity with maximal ratio combining. For this error event, PEP expressions for both channel types yield identical results due to the orthogonality of error event and the coincidental fact that error length [diversity determining parameter in (16)] and number of transmit antennas [diversity determining parameter in (19)] are both equal to two with corresponding eigenvalues of $\lambda_{1,2} = \beta_{1,2} = 4\left(\overline{E}_s/N_0\right)$.

In the following, we focus on the bit-error probability of the example code. The analytical estimates of bit-error probability taking into account error events up to length $v = 2, 3,$ and 4 are plotted in Fig. 1 for the four-state space–time code. Two different cases for one and two receive antennas are considered. The simulation results are also superimposed in the figure for comparison purposes. It is clearly seen that the simulation results agree very well with the analytical estimates. We should note that the ECL of this code is two. Fig. 1 also shows that $v = 2$ provides satisfactory results indicating that most of the dominant error events at the SNR of interest are included. It should be further emphasized that all the estimates, each computed based on different v values, converge for asymptotically high SNR values. These estimates essentially represent a truncation of the infinite series used in calculating the union bound on the bit-error probability. Therefore, if we could take into account all error events (i.e., $v \to \infty$), an upper bound would be obtained. The convergence of our estimates, even at very small values of v, shows that the convergence to the union upper bound is quite fast for the interleaved channel. In Fig. 2, we investigate the same code's performance over a quasi-static Rayleigh fading channel. Similar to Fig. 1, we consider analytical estimates up to $v = 4$. The estimates provide good matches to simulation results taking into account only

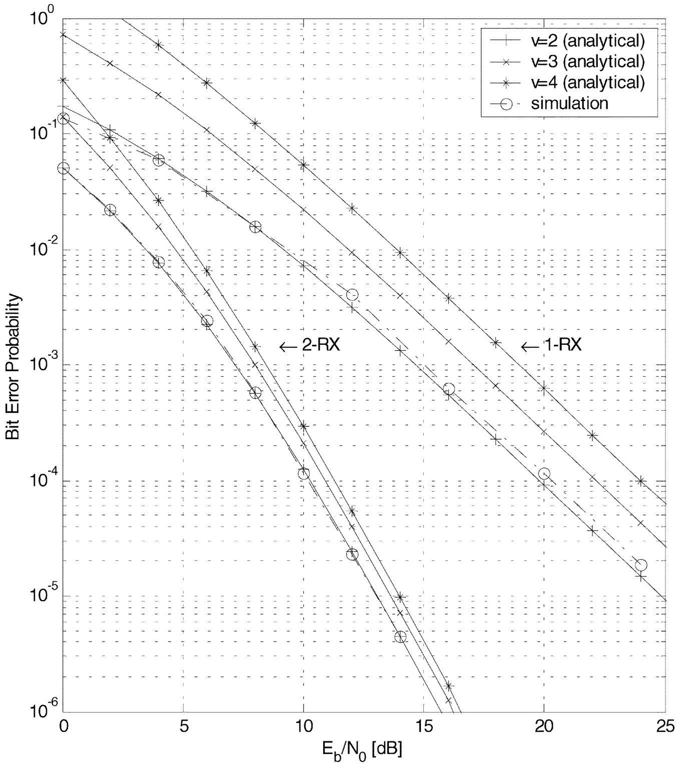

Fig. 2. Performance of the four-state STTC over quasi-static Rayleigh fading channels.

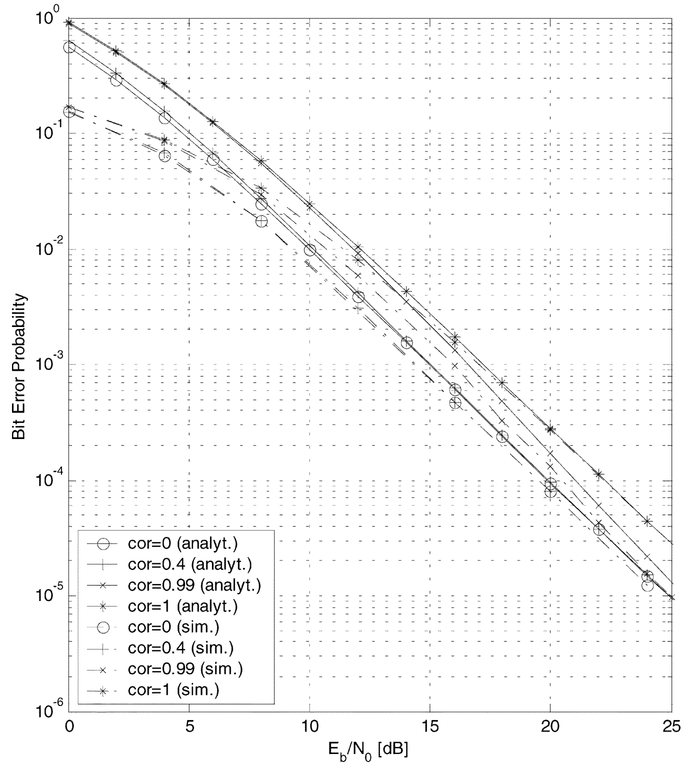

Fig. 3. Effect of spatial correlation on the performance over interleaved Rayleigh fading channels.

a reasonably small number of error events. However, it should be noted that the convergence of estimates to the union upper bound is slower compared to the interleaved channel. This also shows that the union bound becomes looser for a quasi-static channel although it gives tight upper bounds for the interleaved channel.

A comparison of Figs. 1 and 2 reveals that the four-state code gives the same diversity order regardless of the use of interleaving. There is only a small coding gain obtained by the use of interleaving. This can be easily explained based on the code parameters: As shown by (16), the diversity order is determined by $|\varpi'|\,N$. Since the ECL of this code (i.e., the smallest value of $|\varpi|$ over all possible error events) is two, the diversity order achieved is two, assuming one receive antenna is used. As indicated before, the use of multiple transmit antennas does not affect the diversity order over the interleaved channel. On the other hand, for the quasi-static case, the diversity order is determined by γN. Since the codes under investigation are designed to provide full diversity [1], γ is equal to the number of transmit antennas. Therefore, the diversity order is achieved again as two, giving the same order as over the interleaved channel. This is a result of the coincidental fact that ECL is equal to the number of transmit antennas for this specific example.

In the following, we focus on the effect of spatial correlation. Analytical estimates for various values of spatial correlation are shown in Figs. 3 and 4 for interleaved and quasi-static channels, respectively. Simulation results are also superimposed as dashed lines in the figures.

It should be noted that all the error events of length two for this code are orthogonal; therefore, it is not possible to observe the effect of spatial fading correlation through just taking into account error events of length two. v should be chosen at least three in this case. Our results show that the spatial correlation only results in

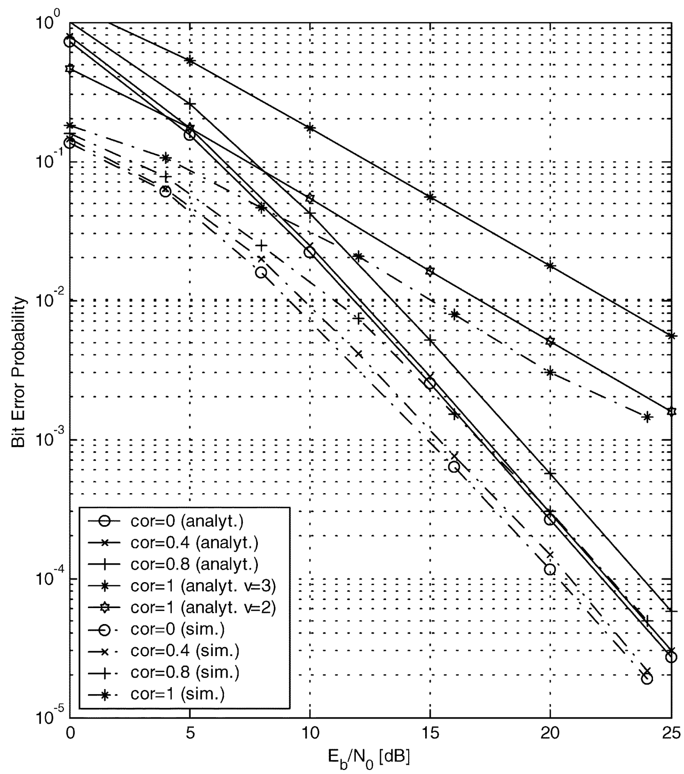

Fig. 4. Effect of spatial correlation on the performance over quasi-static Rayleigh fading channels.

a decrease of coding gain over the interleaved channel. The performance loss up to $\rho = 0.4$ is negligible. The loss is less than 1 dB for $\rho = 0.8$ (not shown in the figure). For the fully correlated case (i.e., $\rho = 1$), there is about a 3-dB loss, however, still preserving the slope of the curve. Another point to note is that the effect of spatial correlation vanishes with increasing SNR. This is observed especially well for the correlation value $\rho = 0.99$. It demonstrates a similar performance to the curve of full correlation (i.e., $\rho = 1$) in the low SNR region; however, with increasing SNR, it performs much better with a tendency to converge to other curves with lower correlation values.

The effect of spatial correlation over the quasi-static channel is much different from the interleaved case. It is observed from Fig. 4 that spatial correlation values of $\rho = 0.4$ and $\rho = 0.8$ result in about 0.5 and 2 dB loss, respectively. In both cases, the diversity order, and therefore, the slope of the curve, is preserved. When ρ increases to one, diversity order is reduced to one, as it can be easily observed from the slope of the curve in Fig. 4. We plot two analytical estimates for this correlation value. One of them is based on error events with length three as for the other values of correlation. However, in the fully correlated case, since some of the error events of length three reduce to length two, which are not orthogonal, (recall that all length-two error events of the code under investigation are orthogonal), we can also obtain an estimate just based on the events with length two for this special case. This estimate is also plotted and seen to provide a better estimate to the corresponding simulation result.

VII. Conclusion

We provide analytical tools for the evaluation of space–time trellis codes operating over Rayleigh fading channels. First, an exact expression for PEP is derived for space–time codes, which essentially constitutes an extension of previous results in [6]–[9] which are limited to symbol-by-symbol interleaved channels where the path gains are assumed constant over one symbol interval. Based on the derived PEP, analytical estimates for bit-error probability are computed taking into account a small number of error events. The effect of spatial correlation is investigated for both quasi-static and symbol-by-symbol interleaved channels. Our results show that the spatial correlation between the transmit antennas over the interleaved channel only results in a change in coding gain, but does not affect the diversity order. For the quasi-static scenario, full correlation reduces the diversity order, but for a large range of correlation values, the performance of space–time code is observed to be robust to spatial correlation.

Acknowledgment

The authors would like to thank the reviewers for their helpful comments, which improved the technical content and presentation of the paper significantly.

References

[1] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, pp. 744–765, Mar. 1998.

[2] A. Naguib, V. Tarokh, N. Seshadri, and A. R. Calderbank, "A space-time coding modem for high-data-rate wireless communications ," *IEEE J. Select. Areas Commun.*, vol. 16, pp. 1459–1478, Oct. 1998.

[3] B. M. Hochwald and T. L. Marzetta, "Unitary space-time modulation for multiple-antenna communications in Rayleigh flat fading," *IEEE Trans. Inform. Theory*, vol. 46, pp. 543–564, Mar. 2000.

[4] J.-C. Guey, M. P. Fitz, M. R. Bell, and W.-Y. Kuo, "Signal design for transmitter diversity wireless communications systems over Rayleigh fading channels," *IEEE Trans. Commun.*, vol. 47, pp. 527–537, Apr. 1999.

[5] D. M. Ionescu, "On space-time code design," *IEEE Trans. Wireless Commun.*, vol. 2, pp. 20–28, Jan. 2003.

[6] M. Uysal and C. N. Georghiades, "Error performance analysis of space-time codes over Rayleigh fading channels," *J. Commun. Networks (JCN)*, vol. 2, no. 4, pp. 351–356, Dec. 2000.

[7] J. G. Proakis, *Digital Communications*, 3rd ed. New York: McGraw-Hill, 1995.

[8] J. K. Cavers and P. Ho, "Analysis of the error performance of trellis coded modulations in Rayleigh fading channels," *IEEE Trans. Commun.*, vol. 40, pp. 74–83, Jan. 1992.

[9] M. Uysal and C. N. Georghiades, "Effect of spatial fading correlation on the performance of space-time codes," *Electron. Lett.*, vol. 37, no. 3, pp. 181–183, Feb. 2001.

[10] ——, "Performance of space-time coded CDMA systems for wireless communication," *Electron. Lett.*, vol. 36, no. 14, pp. 1231–1232, July 2000.

[11] G. L. Turin, "The characteristic function of Hermetian quadratic forms in complex normal random variables," *Biometrika*, pp. 199–201, June 1960.

[12] A. M. Mathai and S. B. Provost, *Quadratic Forms in Random Variables: Theory and Applications*. New York: Marcel Dekker, 1992.

[13] S. H. Jamali and T. L. Ngoc, *Coded Modulation Techniques for Fading Channels*. Norwell, MA: Kluwer, 1994.

[14] R. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1999.

[15] P. Ho and D. K. P. Fung, "Error performance of interleaved trellis coded PSK modulations in correlated Rayleigh fading channels," *IEEE Trans. Commun.*, vol. 40, pp. 1800–1809, Dec. 1992.

[16] E. Malkamaki and H. Leib, "Evaluating the performance of convolutional codes over block fading channels," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1643–1646, July 1999.

Murat Uysal (S'98–M'02) was born in Istanbul, Turkey, in 1973. He received the B.Sc. and M.Sc. degrees in electronics and communication engineering from Istanbul Technical University, Istanbul, Turkey, in 1995 and 1998, respectively, and the Ph.D. degree in electrical engineering from Texas A&M University, College Station, TX, in 2001.

From 1995 to 1998, he worked as a Research and Teaching Assistant in the Communication Theory Group at Istanbul Technical University. From 1998 to 2001, he was a Research Assistant in the Wireless Communication Laboratory, Texas A&M University. He worked as an intern at AT&T Laboratories-Research, Florham Park, NJ, during the fall of 2000. In April 2002, he joined the Department of Electrical and Computer Engineering, University of Waterloo, Canada, as an Assistant Professor. His general research interests lie in digital communications theory with special emphasis on wireless applications. Specific areas include space–time coding, diversity techniques, coding for fading channels, and performance analysis over fading channels.

Costas N. Georghiades (S'82–M'85–SM'90–F'98) received the B.E. degree with distinction from the American University of Beirut, Lebanon, in June 1980, and the M.S. and D.Sc. degrees from Washington University, St. Louis, in May 1983 and May 1985, respectively, all in electrical engineering.

Since September 1985, he has been with the Electrical Engineering Department at Texas A&M University, College Station, TX, where he is a Professor and holder of the Delbert A. Whitaker Endowed Chair. His general interests are in the application of information, communication and estimation theories to the study of communication systems.

Dr. Georghiades is a registered Professional Engineer in Texas. Over the years, he served in editorial positions with the IEEE Transactions on Communications, the IEEE Transactions on Information Theory, the IEEE Journal on Selected Areas in Communications, and the IEEE Communications Letters. He has been involved in organizing a number of conferences, including as Technical Program Chair for the 1997 IEEE Communication Theory Mini Conference and the 1999 IEEE Vehicular Technology Conference. Recently, he served as Chair of the Communication Theory Symposium within Globecom 2001, and as Technical Program Chair for the 2001 Communication Theory Workshop. He currently serves in various committees of the IEEE Communications and Information Theory Societies. He was the recipient of the 1995 Texas A&M University College of Engineering Halliburton Professorship and in 2002 the E. D. Brockett Professorship. From 1997 to 2002 he was the J.W. Runyon Jr. Endowed Professor and in 2002 he became the inaugural recipient of the Delbert A. Whitaker Endowed Chair.