Relay-Assisted Quantum-Key Distribution Over Long Atmospheric Channels

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Abstract—In this paper, we investigate a terrestrial relay-assisted scheme for a free-space quantum key distribution (QKD) system based on BB84 protocol. We consider the deployment of passive relays which simply redirect the qubits to the next relay node or to the receiver without performing any measurement or detection process. Based on a near-field analysis, we derive an upper bound on quantum bit error rate (QBER) of the relay-assisted QKD system. Our results demonstrate that the relay-assisted scheme is able to outperform point-to-point direct transmission for long link ranges in which turbulence effects are particularly degrading.

Index Terms—Free-space optics, quantum-key distribution, relay-assisted systems.

I. INTRODUCTION

CRYPTOSYSTEMS are an indispensable part of modern telecommunication networks to secure the privacy of data transmission and to protect it from electronic copying, cloning, and destroying, and to also deter unauthorized entry into the network. Vast majority of today's cryptosystems are able to offer only computational security within the limitations of conventional computing power and the realization of quantum computers would, for example, make electronic money instantly worthless. Based on the firm laws of quantum mechanics rather than some unproven foundations of mathematical complexity, quantum cryptography provides a radically different solution for key distribution promising unconditional security.

Quantum cryptography builds on a well-known quantum physics rule that no measurement can be taken without perturbing the quantum system, unless the quantum state and the measurement are compatible. It guarantees a provenly-secure communication between a transmitting party (conventionally named as Alice) and a receiving party (conventionally named as Bob) where any eavesdropping attempt by Eve can be detected by them through a public communication during error correction procedure.

An end-to-end quantum cryptosystem includes an initial phase of quantum key distribution (QKD) and a subsequent

one-time pad encryption process [1]. The first QKD protocol proposed by Bennett and Brassard [2] is today widely known as BB84 which uses elementary quantum systems such as polarized photons to transfer secret keys between two points. As an alternative to BB84, Ekert [3] has proposed another protocol which involves the emission of two qubits (quantum bits) from a source to Alice and Bob instead of transmitting a single qubit from Alice to Bob. Assuming that the two emitted qubits from the source are in a maximally entangled state, Ekert's protocol avoids trusting the source which can be targeted by eavesdropping attacks. Despite rather different implementation structures, both protocols guarantee equivalently secured QKD systems. A number of other protocols also exist in the literature [4], but they essentially build on these two main protocols.

In analysis, design, and optimization of QKD systems, a critical issue to consider is the operating environment. The common form of quantum channel which has been intensively studied so far is fiber optic. Although today's telecommunication networks based on optical fibers are very advanced, such channels may not always be available. A cost-effective and easy-to-install alternative to fiber optic is free-space optical (FSO) communications on which this paper focuses. With the potential high-data-rate capacity and wide bandwidth on unregulated spectrum, FSO communications has enjoyed a growing attention in recent years for classical data transmission, see, e.g., [5]–[8]. Its unique properties make it appealing for a number of applications, including last-mile access, metropolitan area network extensions, enterprise/local area network connectivity, fiber backup, back-haul for wireless cellular networks, redundant link, and disaster recovery. With the recent commercialization successes, FSO technology has reached to a certain maturity for classical data transmission and is a reliable alternative to fiber optics in quantum cryptographic applications as well. In contrast to optical fibers, the dispersion (birefringence) effects of FSO links are negligible especially at high transmission windows of atmosphere (e.g., around wavelength of 770 nm). This guarantees the consistency of the propagating photon's polarization and makes FSO particularly attractive for QKD systems.

Previous studies on terrestrial free-space QKD systems have presented analytical approaches to characterize atmospheric effects [9]–[11] and demonstrated some experimental results [12]–[14]. A major disadvantage associated with free-space QKD systems is the performance degradation due to absorption, scattering, diffraction, and turbulent-induced scintillation experienced in atmospheric channels [9]–[14]. This severely limits the range of FSO QKD system.

In this paper, we propose relay-assisted transmission as a powerful method to overcome the range limitations in terrestrial

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free-space QKD systems. Relay-assisted QKD has been originally proposed in [15] for fiber optic QKD systems. In the relaying scheme of [15] called as quantum repeater, long quantum channel is divided into shorter hops in which perfect entangled pairs are created. Connecting these individual entangled pairs through entanglement swapping leads to a single entangled pair which is used for QKD based on Ekert's protocol. The quantum repeaters are able to improve the performance by overcoming the exponential path loss of signal. However, they require using quantum memory and entanglement purification to ensure creation of perfect entangled pairs in the individual hops. In [16], a simpler relaying scheme has been proposed which avoids the use of quantum memory and entanglement purification. This scheme is not anymore able to mitigate the exponential loss of signal, but it still improves the performance of fiber optic QKD systems by suppressing the aggravated effect of detector dark counts experienced in long distance links. Unlike [15], [16], a class of quantum relays called as trusted relays have been further introduced in [17], [18]. Trusted relays are assumed to be sufficiently reliable to have partial or perfect knowledge of the distributing keys. For example, a trusted quantum relay may perform the well-known intercept/resend eavesdropping strategy to direct Alice's transmitted qubits to Bob without endangering the security of the QKD system [17].

Quantum relaying has been also studied over FSO channels, but the current results are mainly restricted for satellite transmission [19], [20]. In [19], Hughes *et al.* have considered a relay-assisted QKD scenario for satellite communication between two terrestrial nodes without line-of-sight path to each other. A trusted relay in a satellite generates secure keys with the two nodes and then reveals the XOR of the two keys [19]. To support the idea, a feasibility analysis is also reported in [19] based on the estimation of key generation capability of QKD system between a ground station and a LEO satellite. In [20], Aspelmeyer *et al.* have studied a similar satellite-based QKD setup using quantum entanglement. They have investigated the link attenuation of surface/satellite-to-satellite systems to demonstrate the feasibility of their idea.

To the best of our knowledge, a detailed investigation of terrestrial relay-assisted free-space QKD systems has not been yet reported. Aiming to fill this research gap, we consider a relay-assisted system employing passive relay(s). The relays are passive in the sense that they simply redirect the qubits without any detection or quantum measurement involved. These relays, which can be implemented by adaptive optics [21]–[25], reconstruct the turbulence-degraded wave-front of the received beam and redirect the resulting collimated beam to the next relay or Bob. We investigate the efficiency of such relay-assisted systems to combat the range limitations in a terrestrial scenario. Based on a near-field analysis [9], we derive an upper-bound on quantum bit error rate (QBER) as a performance criterion of free-space QKD systems and optimize it with respect to the number of relays. Our results demonstrate that, although the relay-assisted QKD scheme increases the average number of background photons collected at Bob, it is able to significantly decrease the photon loss caused by diffraction and turbulence. The proposed scheme therefore outperforms point-to-point counterparts in long ranges.

The rest of paper is organized as follows. In Section II, we introduce the relay-assisted system model and describe the QKD protocol under consideration. In Section III, we derive bounds on sift and error probabilities, and the resulting upper-bound on the QBER of the relay-assisted QKD system. In Section IV, we provide numerical results and discussions based on our derived QBER expression. Finally, we conclude in Section V.

II. SYSTEM MODEL

We consider a relay-assisted free-space QKD system in which the transmitted qubits by Alice travel through K relay nodes before detection by Bob. Conventional amplify-and-forward and detect-and-forward relaying schemes typically deployed in wireless communications are of no use for QKD systems¹. Here, we consider passive relays which simply redirect the qubits to the next relay node or to Bob without performing any measurement or detection process.

Fig. 1 illustrates a schematic diagram of a relay-assisted QKD system which uses BB84 protocol for key distribution. In this system, Alice uses a polarization basis randomly chosen from two certain polarization bases (e.g., $0^\circ/90^\circ$ and $-45^\circ/+45^\circ$) for each qubit. She sends the qubit with a random bit value of 0 or 1 using polarization encoding of photons where the first polarization in each basis (0° and -45°) represents 0 and the second one (90° and $+45^\circ$) represents 1. The relay receives the transmitted optical beam and reconstructs its turbulence-degraded wave-front using adaptive optics before pointing it to the next relay or Bob. Adaptive optics involves the use of a wave-front sensor which estimates the turbulence-induced phase error information at the beginning of each block of key transmission and a deformable mirror which uses this information to correct the atmospheric distortion.

At the receiver side, Bob also chooses a random basis from the two bases using a passive beam splitter. At the outputs of the beam splitter, two polarization detection units measure the quantum state of the possibly coming photon based on the two different bases. Each of these units includes a polarizing beam splitter (PBS) to decide between two orthogonal polarization states of the corresponding basis and two single-photon Geiger-mode avalanche photodiodes (APDs) at the outputs of the PBS for photon count. To avoid uncorrelated measurement results due to incompatibility of the chosen bases, Alice and Bob construct the secure key only based on the qubits received at the *sift* events. Sift events correspond to the bit intervals in which exactly one of the APDs registers a count and both Alice and Bob have chosen the same basis. According to BB84 protocol, Alice and Bob can identify the sift events by exchanging information in a public communication channel. After identifying the sifted qubits, Alice and Bob follow a standard set of operations to identify and correct errors which have occurred in the sifted bits. These errors are caused by dark counts or background noise as well as by potential eavesdropper's intervention. Then, Alice and Bob perform a procedure called *privacy amplification* to prevent Eve from keeping useful information about the key

¹No amplification can be performed at relay nodes without perturbing the quantum system [4]. Furthermore, since the relays are not necessarily trusted, neither measurements nor detection can be performed at the relay nodes.

Fig. 1. Schematic diagram of the relay-assisted QKD system with a single relay ($K = 1$).

and finally establish a shared one-time pad key to use for secure communication [4]. It is worth emphasizing that any eavesdropping attempt which takes place either at the relay nodes or at any other point within the FSO link can be detected by Alice and Bob during the error correction process of the BB84 protocol.

For each qubit, Alice generates a polarized optical pulse with an average photon number of n_s which is encoded with the corresponding polarization state of the qubit for a randomly chosen basis. Due to the atmospheric effects (i.e., diffraction, atmospheric turbulence, and absorption-and-scattering loss), the i th relay node ($i = 1, 2, \dots, K$) collects only a random fraction γ_i of the transmitted photons in the i th hop. The relay then redirects the received photons by pointing the light beam to the next relay node (or Bob) without performing any amplification/detection process. At the destination side, Bob collects a fraction γ_{K+1} of the transmitted photons from the last relay node which corresponds to an overall fraction² $\gamma = \gamma_1 \gamma_2 \dots \gamma_{K+1}$ of the originally transmitted photons from Alice.

In the i th hop ($i = 1, 2, \dots, K + 1$), a spatial beam pattern $\xi_i(\vec{\rho})$ is assumed to be transmitted from a circular pupil R_i with diameter d_i located in the $z = 0$ plane.³ A field pattern $\xi'_i(\vec{\rho}')$ is collected within the pupil R'_i with diameter d'_i in the $z = L_i$ plane where L_i is the length of i th hop. The received field pattern by the i th relay ($i = 1, 2, \dots, K$) or Bob ($i = K + 1$) can be expressed as [9]

$$\xi'_i(\vec{\rho}') = \int_{R_i} \xi_i(\vec{\rho}) h_i(\vec{\rho}, \vec{\rho}') e^{-aL_i/2} d\vec{\rho} \quad (1)$$

where a is the extinction coefficient which determines the loss due to absorption and scattering. In (1), $h_i(\vec{\rho}, \vec{\rho}')$ denotes the Green's function for atmospheric propagation through clear turbulent air in the i th hop and is given by [9]

$$h_i(\vec{\rho}, \vec{\rho}') = \frac{e^{jkL_i + jk|\vec{\rho} - \vec{\rho}'|^2/2L_i}}{j\lambda L_i} e^{j\chi(\vec{\rho}, \vec{\rho}') + j\phi(\vec{\rho}, \vec{\rho}')} \quad (2)$$

²We assume negligible optics losses within the relay nodes.

³Without loss of generality, we assume that the coordinate system is properly transformed for each hop.

where λ is the wavelength and $k = 2\pi/\lambda$ is the wave number. In (2), $\chi(\vec{\rho}, \vec{\rho}')$ and $\phi(\vec{\rho}, \vec{\rho}')$ are, respectively, the stochastic log-amplitude and phase fluctuation terms describing scintillation caused by atmospheric turbulence. The strength of atmospheric turbulence is characterized by the variance of log-amplitude fluctuation σ_χ^2 which is a function of link length L_i , wave number k , and refractive index structure constant C_n^2 . It is given by [26]

$$\sigma_\chi^2(d) = 0.124k^{7/6}C_n^2L_i^{11/6}. \quad (3)$$

The fraction parameter [9] of the i th hop ($i = 1, 2, \dots, K + 1$) is expressed in terms of transmit and receive beam patterns as

$$\gamma_i = \frac{\int_{R'_i} |\xi'_i(\vec{\rho}')|^2 d\vec{\rho}'}{\int_{R_i} |\xi_i(\vec{\rho})|^2 d\vec{\rho}}. \quad (4)$$

We assume that the relay and destination nodes collect an average number of n_{B0} background photons per polarization. Taking into account the background photons redirected by relays, the total average number of background photons collected at Bob's receiver per polarization can be expressed as⁴

$$\begin{aligned} n_B &= n_{B0}(1 + \gamma_{K+1} + \gamma_K \gamma_{K+1} + \dots + \gamma_2 \dots \gamma_{K+1}) \\ &= n_{B0} \left[1 + \sum_{i=2}^{K+1} \prod_{j=i}^{K+1} \gamma_j \right]. \end{aligned} \quad (5)$$

Besides the background noise, each of the single-photon detectors (APDs) at Bob's receiver registers an average number of n_D dark counts.

⁴Equation (5) overestimates the effect of redirected background photons for the relay-assisted system since the associated path loss is assumed to be as low as the path loss of the transmitted collimated beam.

III. PERFORMANCE BOUNDS FOR THE RELAY-ASSISTED FREE-SPACE QKD SYSTEM

In this section, we investigate the performance of the relay-assisted free-space QKD system through the derivation of an upper-bound on QBER. QBER is given by [4]

$$\text{QBER} = \frac{\Pr(\text{error})}{\Pr(\text{sift})} \quad (6)$$

where $\Pr(\text{sift})$ and $\Pr(\text{error})$ are probabilities of sift and error, respectively. Using the results from [9], we can express the conditional sift and error probabilities of the relay-assisted QKD system (conditioned on $\mathbf{V}_\gamma = \{\gamma_1, \gamma_2, \dots, \gamma_{K+1}\}$) as

$$\Pr(\text{sift}|\mathbf{V}_\gamma) = \eta \left(\frac{n_S \gamma}{2} + 2n_N \right) \cdot \exp[-\eta(n_S \gamma + 4n_N)] \quad (7)$$

$$\Pr(\text{error}|\mathbf{V}_\gamma) = \eta n_N \exp[-\eta(n_S \gamma + 4n_N)] \quad (8)$$

where η is the quantum efficiency of Geiger-mode APDs and $n_N \triangleq n_B/2 + n_D$ denotes the average number of noise counts at each of Bob's detectors.

To obtain unconditional probabilities of sift and error, we first need to statistically characterize $\mathbf{V}_\gamma$. We use the singular value decomposition of $h_i(\vec{\rho}, \vec{\rho}')$, $i = 1, 2, \dots, K+1$, which is given by [9], [27]

$$h_i(\vec{\rho}, \vec{\rho}') = \sum_{n=1}^{\infty} \sqrt{\mu_i^n} \phi_i^n(\vec{\rho}') \Phi_i^{n*}(\vec{\rho}) \quad (9)$$

where μ_i^n 's are the eigenvalues which satisfy $1 \geq \mu_i^1 \geq \mu_i^2 \geq \mu_i^3 \geq \dots \geq 0$. $\{\Phi_i^n(\vec{\rho})\}$ and $\{\phi_i^n(\vec{\rho}')\}$ are, respectively, the input and output eigenfunction vectors which constitute complete orthonormal sets on R_i and R'_i . Let μ_i denote the largest eigenvalue in (9) (i.e., $\mu_i \triangleq \mu_i^1$) and $\Phi_i(\vec{\rho})$ denote the corresponding input eigenfunction (i.e., $\Phi_i(\vec{\rho}) \triangleq \Phi_i^1(\vec{\rho})$). Using (1), (4), and (9), an upper bound on the fraction parameter γ_i can be obtained as [9]

$$\gamma_i \leq \mu_i e^{-aL_i} \quad (10)$$

with equality when $\bar{\xi}_i(\vec{\rho}) = \Phi_i(\vec{\rho})$. Here, $\bar{\xi}_i(\vec{\rho})$ is the normalized transmit beam pattern of the i th hop which can be expressed as

$$\bar{\xi}_i(\vec{\rho}) = \frac{\xi_i(\vec{\rho})}{\sqrt{\int_{R_i} |\xi_i(\vec{\rho})|^2 d\vec{\rho}}}. \quad (11)$$

Under atmospheric turbulence conditions which exhibit a slowly-varying nature, Alice and relay nodes can employ adaptive optics to generate the optimum transmit field pattern as

$$\xi_i(\vec{\rho}) = \left[\sqrt{\int_{R_i} |\xi_i(\vec{\rho})|^2 d\vec{\rho}} \right] \Phi_i(\vec{\rho}) \quad (12)$$

which guarantees that the equality in (10) is achieved yielding the optimum fraction parameter (i.e., maximum power transfer). Let μ denote the product of the individual largest eigenvalues, i.e., $\mu = \mu_1 \mu_2 \dots \mu_{K+1}$ ($0 \leq \mu \leq 1$). The optimum overall fraction parameter can be then written as $\gamma = \mu e^{-aL}$ where $L = L_1 + L_2 + \dots + L_{K+1}$. Unfortunately, for near-field propagation, a statistical description of μ_i is not available in the literature. However, note that $E\{\mu_i\}$ is the maximum achievable average power transfer over the i th hop obtained through the use of optimum beam pattern. Therefore, it can be simply lower bounded by the average power transfer of any deterministic beam pattern, such as in [9], [27], as

$$E\{\mu_i\} \geq \hat{\mu}_i \equiv \frac{8\sqrt{D_f}}{\pi} \int_0^1 e^{-(D_i(d_i x)/2)} J_1(4x\sqrt{D_f}) \times (\cos^{-1}(x) - x\sqrt{1-x^2}) dx \quad (13)$$

where $J_1(\cdot)$ is the first-order Bessel function of the first kind, $D_i(\cdot)$ is the i th hop's spherical-wave wave structure function given by $D_i(\nu) = 1.09k^2 C_n^2 L_i \nu^{5/3}$, and D_f is the Fresnel number product of the R_i and R'_i pupils given by $D_f = ((\pi d_i d'_i)/4\lambda L_i)^2$.

A. Bound on the Sift Probability

Let $\mathbf{V}_\mu = \{\mu_1, \mu_2, \dots, \mu_{K+1}\}$ denote the vector of the largest eigenvalues (within each hop) and define $Z(\mathbf{V}_\mu)$ as a function of $\mathbf{V}_\mu$ as

$$\begin{aligned} Z(\mathbf{V}_\mu) &= \eta(n_S \gamma + 4n_N) \\ &= \eta n_S e^{-aL} \prod_{j=1}^{K+1} \mu_j + 4\eta n_D \\ &\quad + 2\eta n_{B0} \left[1 + \sum_{i=2}^{K+1} \prod_{j=i}^{K+1} e^{-aL_j} \mu_j \right]. \end{aligned} \quad (14)$$

Using (14), we can rewrite (7) for the case of optimum beam transmission as

$$\Pr(\text{sift}|\mathbf{V}_\mu) = \frac{Z(\mathbf{V}_\mu)}{2} e^{-Z(\mathbf{V}_\mu)} = \frac{(\alpha y + \beta)}{2} e^{-(\alpha y + \beta)} \quad (15)$$

where α , β , and y are given by⁵

$$\begin{aligned} \alpha &= Z(\mathbf{V}_\mu = \mathbf{1}) - Z(\mathbf{V}_\mu = \mathbf{0}) \\ &= \eta \left[n_S e^{-aL} + 2n_{B0} \left[\sum_{i=2}^{K+1} \prod_{j=i}^{K+1} e^{-aL_j} \mu_j \right] \right] \end{aligned} \quad (16)$$

$$\beta = Z(\mathbf{V}_\mu = \mathbf{0}) = \eta(2n_{B0} + 4n_D) \quad (17)$$

$$y = \frac{Z(\mathbf{V}_\mu) - \beta}{\alpha}. \quad (18)$$

Averaging (15) over y , we obtain the sift probability as

$$\Pr(\text{sift}) = \int_0^1 \frac{(\alpha y + \beta)}{2} e^{-(\alpha y + \beta)} p(y) dy \quad (19)$$

⁵0 and 1 denote all-zero and all-one vectors, respectively,

where $p(y)$ is the probability density function (pdf) of y . Defining $f(y) \triangleq (\alpha y/2 + \beta/2)e^{-(\alpha y + \beta)}$ and noting that it is a concave function for $0 \leq \alpha y + \beta < 2$, we can lower bound (19) as

$$\begin{aligned} \Pr(\text{sift}) &\geq \int_0^1 [f(0)(1-y) + f(1)y]p(y)dy \\ &= f(0)(1 - E\{y\}) + f(1)E\{y\}. \end{aligned} \quad (20)$$

Using (13), (14), and (18), and the fact that the elements of $\mathbf{V}_\gamma$ (and therefore the elements of $\mathbf{V}_\mu$) are statistically independent, $E\{y\}$ can be bounded as

$$\begin{aligned} E\{y\} &\geq \hat{y} \\ &\triangleq \frac{n_S e^{-aL} \prod_{j=1}^{K+1} \hat{\mu}_j + 2n_{B0} \left[\sum_{i=2}^{K+1} \prod_{j=i}^{K+1} e^{-aL_j} \hat{\mu}_j \right]}{n_S e^{-aL} + 2n_{B0} \left[\sum_{i=2}^{K+1} \prod_{j=i}^{K+1} e^{-aL_j} \right]}. \end{aligned} \quad (21)$$

Inserting (21) in (20) and noting that $f(y)$ is increasing for $0 \leq \alpha y + \beta < 1$, we have a lower bound on the sift probability as

$$\begin{aligned} \Pr(\text{sift}) &\geq f(0)(1 - \hat{y}) + f(1)\hat{y} \\ &= \left(\frac{\beta}{2}\right) e^{-\beta} [1 - \hat{y}] + \left(\frac{\alpha}{2} + \frac{\beta}{2}\right) e^{-(\alpha + \beta)\hat{y}} \end{aligned} \quad (22)$$

under the practically reasonable condition of

$$\eta(n_S e^{-aL} + 4n_N) < 1.$$

B. Bound on the Error Probability

Ignoring the effects of diffraction and turbulence on the redirected background photons (from the relays), a pessimistic noise count at each of Bob's detectors (i.e., $\hat{n}_N = n_N | \mathbf{v}_\mu = \mathbf{1}$) can be written as

$$\hat{n}_N = \frac{n_{B0}}{2} \left[1 + \sum_{i=2}^{K+1} \prod_{j=i}^{K+1} e^{-aL_j} \right] + n_D \quad (23)$$

which obviously satisfies $\hat{n}_N \geq n_N$. Therefore, we can bound (8) as

$$\Pr(\text{error} | \mathbf{V}_\gamma) \leq \eta \hat{n}_N e^{-\eta(n_S \gamma + 4\hat{n}_N)}. \quad (24)$$

Inserting $\gamma = \mu e^{-aL}$ in (24) and averaging the resulting expression over μ , we have

$$\Pr(\text{error}) \leq \int_0^1 \eta \hat{n}_N e^{-\eta(n_S \mu e^{-aL} + 4\hat{n}_N)} p(\mu) d\mu \quad (25)$$

where $p(\mu)$ is the pdf of μ . Defining $g(\mu) \triangleq \eta \hat{n}_N e^{-\eta(n_S \mu e^{-aL} + 4\hat{n}_N)}$ and noting that it is a convex function, we can further upper bound (25) as

$$\Pr(\text{error}) \leq \int_0^1 [g(0)(1 - \mu) + g(1)\mu] P(\mu) d\mu$$

$$\begin{aligned} &= g(0)(1 - E\{\mu\}) + g(1)E\{\mu\} \\ &\leq g(0)(1 - \hat{\mu}) + g(1)\hat{\mu} \end{aligned} \quad (26)$$

where we define $\hat{\mu} = \hat{\mu}_1 \hat{\mu}_2 \cdots \hat{\mu}_{K+1}$. The first bound in (26) comes from the convexity of $g(\mu)$. The last bound results from the bound in (13) assuming that μ_i 's are statistically independent and noting that $g(\mu)$ is a decreasing function. Expanding (26), we obtain

$$\Pr(\text{error}) \leq \eta \hat{n}_N e^{-\eta 4\hat{n}_N} [1 - \hat{\mu}] + \eta \hat{n}_N e^{-\eta(n_S e^{-aL} + 4\hat{n}_N)} \hat{\mu}. \quad (27)$$

Replacing (22) and (27) in (6), we obtain an upper bound on QBER as

$$\text{QBER} \leq \frac{2\eta \hat{n}_N e^{-\eta 4\hat{n}_N} [(1 - \hat{\mu}) + e^{-\eta n_S e^{-aL}} \hat{\mu}]}{\beta e^{-\beta} (1 - \hat{y}) + (\alpha + \beta) e^{-(\alpha + \beta)\hat{y}}}. \quad (28)$$

It is important to note that, although at the beginning of this derivation we consider optimum beam shaping in (10) to achieve the equality, the derived bound in (28) is valid for non-optimized transmission. Because the lower-bound on average μ_i given in (13) is based on the assumption of a deterministic beam pattern rather than the optimum one [9].

Finally, as a benchmark to demonstrate the efficiency of the proposed relaying scheme in compensating the degrading effects of atmospheric turbulence, we consider a QKD system operating over non-turbulent air. The exact QBER of such a QKD system is given by [9]

$$\text{QBER} = \frac{2n_N}{n_S \mu_0 e^{-aL_{AB}} + 4n_N} \quad (29)$$

where L_{AB} is the length of line-of-sight link connecting Alice and Bob and μ_0 is the largest eigenvalue of the singular value decomposition of vacuum-propagation Green's function given in [28].

IV. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we demonstrate the performance of relay-assisted QKD scheme under consideration and compare its performance with point-to-point direct transmission. We assume $\lambda = 0.77 \mu\text{m}$, $C_n^2 = 1 \times 10^{-15} \text{ m}^{-2/3}$, and operation in clear weather conditions with visibility of 10 km which corresponds to the absorption and scattering loss of about 2 dB/km. Under the assumption that proper temporal, spectral, and spatial filters are employed, a constant average number of background count, $n_{B0} = 10^{-4}$ is assumed [12]. Bob's APD detectors are operated at the detection efficiency of $\eta = 0.5$ with average dark count of $n_D = 10^{-6}$ [9]. We also assume that all the receive and transmit pupils have the same diameter d and the relay nodes are located equidistant along the path connecting Alice and Bob, i.e., $L_{AB} = L$.

Before we illustrate QBER performance, we study the behavior of the overall fraction of collected photons at Bob's receiver, i.e., γ . We normalize γ by the total amount of absorption and scattering loss experienced in the direct path from Alice to Bob, i.e., $e^{-aL_{AB}}$. Based on the assumption made on the relay locations, we have $\gamma/e^{-aL_{AB}} = \mu e^{-aL}/e^{-aL_{AB}} = \mu$. Fig. 2 plots the lower bound on the average of μ (i.e., $\hat{\mu}$) calculated using (13) for a relay-assisted QKD system with pupil diameter

Fig. 2. Normalized fraction of collected photons at Bob's receiver in relay-assisted QKD system for different number of relays ($d = 10$ cm).

of $d = 10$ cm operating over turbulent atmosphere. We consider relay-assisted systems with different number of relays, i.e., $K = 1, 2$, and 3 . The results for direct QKD systems operating over turbulent (i.e., $\hat{\mu}_t$, $K = 0$) and non-turbulent (i.e., μ_0 given in [28]) atmosphere are further included as benchmarks. Comparing the performance curves of over the turbulent and non-turbulent atmosphere for the direct systems in Fig. 2, we observe that atmospheric turbulence degrades the performance of QKD systems particularly over the long links where stronger turbulence effects are experienced. For example, the average number of photons collected over a 10-km non-turbulent atmospheric path is more than three times the average number of collected photons in the corresponding turbulent case. It is obvious from Fig. 2 that, similar to the classical multi-hop FSO system [8], the relay-assisted QKD scheme takes advantage of the resulting shorter hops with weaker turbulence effects and improves the performance of the QKD system. For example, deployment of two relays would be sufficient to compensate the decrease in the average number of collected photons caused by turbulence for a 10-km link.

Fig. 3 demonstrates the upper bound on QBER performance of the relay-assisted and direct QKD systems operating over turbulent atmosphere as a function of the link range ($L_{AB} = L$) which are calculated through (28). We assume $d = 10$ cm and $n_s = 1$. As a benchmark, we include the QBER performance of direct QKD system over non-turbulent atmosphere calculated through (29). It is observed from Fig. 3 that the relay-assisted scheme improves the QBER performance of the links longer than 5.5 km. For example, for a link with the length of 10 km, the presence of turbulence causes 193% increase in QBER of the direct QKD system. However, this degradation is drastically reduced to only 26% for a triple-hop ($K = 2$) relay-assisted QKD system.

Although the triple-hop relay-assisted QKD system can significantly improve the QBER performance, it cannot achieve the QBER performance of the direct non-turbulent case as it might be expected from Fig. 2. This is as a result of the accumulation

Fig. 3. QBER of the relay-assisted QKD system for different number of relays ($d = 10$ cm, $n_s = 1$).

Fig. 4. QBER of the relay-assisted QKD system versus the number of relays with different pupil diameters ($L = 10$ km, $n_s = 1$).

of background noise redirected from relays to Bob's receiver.⁶ In other words, there are two reverse effects which need to be considered: the relay-assisted scheme increases the average number of collected photons coming from Alice, but it also results in an increase of the average number of background photons at Bob's receiver. It is therefore important to determine the optimum number of relays (in the sense of QBER minimization) to be deployed.

In Fig. 4, we demonstrate the upper bound on QBER of the relay-assisted QKD system with $n_s = 1$ versus the number of relays. We fix the link length as 10 km and consider different pupil diameters i.e., $d = 6$ cm, 10 cm, and 20 cm. We find out that four, two and one relays are sufficient to minimize QBER, respectively, for $d = 6$ cm, 10 cm, and 20 cm. It is observed that the optimum number of relays increases as the pupil size decreases. This comes from the fact that for the system with

⁶Note that, in our analysis, this effect has been exaggerated in derivation of upper bound on QBER of the relay-assisted system, cf. (5) and (23).

Fig. 5. QBER of the relay-assisted QKD system versus n_s for different link lengths ($d = 10$ cm).

smaller pupil sizes, shorter hops are required for effective compensation of the diffraction and turbulence effects.

In Fig. 5, we demonstrate the upper bound on QBER of the relay-assisted QKD system as a function of the average number of transmitted photons by Alice, n_s . We consider $L = 6$ km and 10 km and assume $d = 10$ cm. Note that the QBER performance improvements demonstrated in Figs. 3 and 4 were obtained for a fixed value of $n_s = 1$. From Fig. 5, we observe that decreasing the value of n_s does not considerably affect the amount of QBER improvement and, therefore, similar conclusions can be also made for these lower values of n_s .

As a final remark on our performance results, we need to re-emphasize that adaptive optics is required to reconstruct the turbulence-degraded received wave-front in the proposed system. Different adaptive optics systems with fully [21]–[23] and partially [24], [25] compensation can be used for this purpose. In a fully compensated adaptive optics system, the scale of the operating elements (sub-aperture of the wave-front sensor and the actuator of the deformable mirror) matches the scale of the atmospheric correlation length which is inversely related to the link length [23], [24]. Since the proposed relay-assisted system divides the link range into shorter hops, adaptive optics can be implemented relatively easier with larger, therefore fewer, elements. Furthermore, we note that the atmospheric coherence time is inversely related to the link length [26]. Therefore, longer atmospheric coherence time can be experienced in the shorter hops decreasing the rate of required updates in the adaptive optics system.

V. CONCLUSION

In this paper, we proposed a terrestrial relay-assisted free-space QKD system for long links operating over turbulent atmosphere. Based on a near-field analysis, we derived an upper bound on QBER of the proposed system and presented extensive numerical results on the performance as a function of various system parameters including the number of relays, link range, pupil diameter, etc. Although the relay-assisted scheme results

in an increase of the average number of background photons collected at Bob's receiver, it is able to significantly decrease the photon loss caused by diffraction and turbulence. The net gain is an overall performance improvement particularly for long link ranges.

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