Applied Mathematics Letters 23 (2010) 356–360

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Non-existence and existence of localized solitary waves for the two-dimensional long-wave–short-wave interaction equations

H. Borluk, H.A. Erbay *, S. Erbay

Department of Mathematics, Isik University, 34980 Sile-Istanbul, Turkey

ARTICLE INFO

Article history:
Received 3 July 2009
Received in revised form 25 October 2009
Accepted 28 October 2009

Keywords:
Long-wave–short-wave interaction equations
Non-existence of solitary waves
Existence of solitary waves

ABSTRACT

In this study, we establish the non-existence and existence results for the localized solitary waves of the two-dimensional long-wave–short-wave interaction equations. Both the non-existence and existence results are based on Pohozaev-type identities. We prove the existence of solitary waves by showing that the solitary waves are the minimizers of an associated variational problem.

1. Introduction

In this study we mainly establish the non-existence results and the existence results for the localized solitary waves of the two-dimensional long-wave–short-wave interaction (LSI) equations:

$$\mathrm{i}\phi_t + \phi_{xx} = \phi u_x, \tag{1}$$

$$u_{tx} + \gamma u_{yy} = -(|\phi|^2)_x, \tag{2}$$

where $\gamma \in \mathbb{R}$, $\phi(x, y, t)$ is a complex-valued function, $u(x, y, t)$ is real, $(x, y) \in \mathbb{R}^2$, $t \geq 0$ and subscripts refer to partial derivatives. The LSI equations describe the interaction between high-frequency and low-frequency waves near the long-wave–short-wave resonance where the group velocity of short-waves is equal to the phase velocity of long-waves. The constant γ measures transverse effects in the y-direction for waves propagating essentially in the x-direction. The LSI equations arise in various contexts such as water waves [1, p. 214], geometric optics [2] and elastic waves [3]. The well-posedness of the Cauchy problem associated with the LSI equations has been established in [2].

Existence of solitary waves which are localized traveling waves has been a topic of interest in the study of nonlinear dispersive wave equations. In the absence of y-dependence, (1) and (2) reduce to their one-dimensional form which has been derived in various fields of physics (see, e.g. [4–7]). The existence of the one-dimensional solitary wave solutions for the LSI equations has been established in [8]. In fact, the one-dimensional solitary wave solutions of the LSI equations are given explicitly by

$$\phi(x, t) = 2^{1/2}c^{3/2}\,\mathrm{sech}[c(x + ct)] \exp\left\{\mathrm{i}\left(-\frac{c}{2}x + \frac{3}{4}c^2 t\right)\right\}, \tag{3}$$

$$u_x(x, t) = -2c^2\,\mathrm{sech}^2[c(x + ct)] \tag{4}$$

* Corresponding author. Tel.: +90 216 528 7115; fax: +90 216 712 1474.
E-mail addresses: hborluk@isikun.edu.tr (H. Borluk), erbay@isikun.edu.tr (H.A. Erbay), serbay@isikun.edu.tr (S. Erbay).

0893-9659/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2009.10.010

with $c > 0$ [4]. It is therefore natural to ask whether the localized solitary wave solutions of the LSI equations in the two-dimensional case exist as well. In the present work, we resolve the issue of existence of two-dimensional solitary waves for the LSI equations.

We consider two-dimensional localized solitary wave solutions of (1) and (2) in the form

$$\phi(x,t) = \mathrm{e}^{\mathrm{i}(\omega t+\beta x)}\Phi(x+ct, y+bt), \quad u(x,t) = U(x+ct, y+bt) \tag{5}$$

where $\omega, c, b, \beta \in \mathbb{R}$ and $\Phi \in H^1(\mathbb{R}^2)$, and $\nabla U \in L^2(\mathbb{R}^2)$ are real-valued functions with $\Phi(\xi,\eta)$, $\nabla\Phi(\xi,\eta)$, $\nabla U(\xi,\eta) \to 0$ as $\xi^2+\eta^2 \to \infty$. Here H^1 is the usual Sobolev space on $\mathbb{R}^2$ and L^2 is the Hilbert space equipped with the usual inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2$, respectively. In Theorem 2 of Section 2, we are able to prove the non-existence of two-dimensional localized solitary wave solutions of the LSI equations if $\gamma \le 0$ or $\gamma c \le 0$ or $\omega \le -c^2/4$. In Theorem 4 of Section 3, we prove the existence of two-dimensional localized solitary waves of the LSI equations when $\gamma > 0$, $\gamma c > 0$ and $\omega > -c^2/4$. The way to show the non-existence and the existence of solitary wave solutions passes from Pohozaev-type identities. The problem of proving the existence of solitary wave solutions for the LSI equations reduces to proving the existence of nontrivial solutions for a single second-order partial differential equation with a nonlocal term.

2. Non-existence of solitary wave solutions

Substituting (5) into (1) and (2) we obtain

$$\Phi_{xx} - (\omega+\beta^2)\Phi = \Phi U_x, \tag{6}$$

$$(c+2\beta)\Phi_x + b\Phi_y = 0, \tag{7}$$

$$cU_{xx} + bU_{xy} + \gamma U_{yy} = -(\Phi^2)_x. \tag{8}$$

By (7) Φ is constant along the characteristics lines $bx-(c+2\beta)y =$ constant. For a function $\Phi \in H^1(\mathbb{R}^2)$ this could be accomplished if and only if $\beta = -c/2$ and $b = 0$. Then (5) becomes

$$\phi(x,t) = \mathrm{e}^{\mathrm{i}(\omega t-\frac{c}{2}x)}\Phi(x+ct, y), \quad u(x,t) = U(x+ct, y). \tag{9}$$

If $\tilde{\omega} = \omega + c^2/4$, then (6) and (8) reduce to

$$\Phi_{xx} - \tilde{\omega}\Phi = \Phi U_x, \tag{10}$$

$$cU_{xx} + \gamma U_{yy} = -(\Phi^2)_x. \tag{11}$$

We now establish Pohozaev-type identities from which one can prove the non-existence of solitary wave solutions.

Lemma 1. *Let (ϕ, u) be a solitary wave solution of (1) and (2) in the form (9) with $\Phi \in H^1(\mathbb{R}^2)$, $\nabla U \in L^2(\mathbb{R}^2)$. Then Φ and U must satisfy*

$$\int_{\mathbb{R}^2} [c(U_x)^2 - \gamma(U_y)^2]\mathrm{d}x\mathrm{d}y = 0, \tag{12}$$

$$\int_{\mathbb{R}^2} [(\Phi_x)^2 - \tilde{\omega}\Phi^2]\mathrm{d}x\mathrm{d}y = 0, \tag{13}$$

$$\int_{\mathbb{R}^2} [(\Phi_x)^2 - \gamma(U_y)^2]\mathrm{d}x\mathrm{d}y = 0. \tag{14}$$

Proof. These identities are derived by direct computations. Firstly, we multiply (10) by $x\Phi_x$, integrate the resulting equation over $\mathbb{R}^2$ and use (11). After two uses of integration by parts we get

$$\int_{\mathbb{R}^2} \left[(\Phi_x)^2 - \tilde{\omega}\Phi^2 + \frac{c}{2}(U_x)^2 - \frac{\gamma}{2}(U_y)^2\right]\mathrm{d}x\mathrm{d}y = 0. \tag{15}$$

Similarly, we multiply (10) by $y\Phi_y$, integrate the resulting equation over $\mathbb{R}^2$ and use (11). After several integrations by parts we obtain

$$\int_{\mathbb{R}^2} \left[(\Phi_x)^2 + \tilde{\omega}\Phi^2 - \frac{c}{2}(U_x)^2 - \frac{3\gamma}{2}(U_y)^2\right]\mathrm{d}x\mathrm{d}y = 0. \tag{16}$$

Finally, we multiply (10) by Φ, integrate the resulting equation over $\mathbb{R}^2$ and use (11). After several integrations by parts we obtain

$$\int_{\mathbb{R}^2} [(\Phi_x)^2 + \tilde{\omega}\Phi^2 - c(U_x)^2 - \gamma(U_y)^2]\mathrm{d}x\mathrm{d}y = 0. \tag{17}$$

Subtracting (16) from (17) yields (12). Using (12) in (15) gives (13). Finally, substituting (12) and (13) into (17) gives (14). This completes the proof. □

Theorem 2. *The LSI equations, (1) and (2), have no nontrivial solitary wave solution of the form (9) if $\gamma \leq 0$ or $\gamma c \leq 0$ or $\omega \leq -c^2/4$.*

Proof. The identities (12)–(14) imply that $\Phi \equiv 0$ and that $U \equiv$ constant if $\gamma \leq 0$ or $\gamma c \leq 0$ or $\omega \leq -c^2/4$. This shows that no solitary wave solutions of the form (9) to (1) and (2) exist under any one of the conditions. □

Remark 1. It is worth adding further that the result obtained in Theorem 2 depends on the sign of γ which characterizes transverse dispersion. When $\gamma < 0$, our non-existence result in Theorem 2 covers the whole parameter range for c and it does not leave a gap in the parameter range where the non-existence result could not be established. That is, (1) and (2) have no nontrivial standing wave solutions in the form $\phi = \mathrm{e}^{\mathrm{i}\omega t}\Phi(x, y)$, $u = U(x, y)$ which corresponds to the case $c = 0$ in (9), as well as right-going and left-going solitary waves which corresponds to $c > 0$ and $c < 0$, respectively. When $\gamma > 0$, (1) and (2) have no left-going solitary waves and, if $\omega \leq -c^2/4$, no right-going solitary waves. Notice that there is no non-existence result for the case $\gamma > 0$, $c > 0$ and $\omega > -c^2/4$.

Remark 2. The non-existence result in Theorem 2 is consistent with the line-solitary wave solutions given in [3] and the solitary wave solutions (3)-(4). These solitary wave solutions are not in $L^2(\mathbb{R}^2)$ and therefore cannot be taken into account within the context of Theorem 2.

Remark 3. When we consider the one-dimensional case $\Phi = \Phi(x)$, $U = U(x)$, Eqs. (10) and (11) reduce to

$$\Phi'' - \tilde{\omega}\Phi + \frac{1}{c}\Phi^3 = 0, \tag{18}$$

where we have assumed that $\Phi, U' \to 0$ as $x \to \pm\infty$. We note that the solitary wave solution (3) is a solution of (18) for the special case $\tilde{\omega} = c^2$.

3. Existence of solitary wave solutions

Assume that $\gamma > 0$, $c > 0$ and $\omega > -c^2/4$. We first convert Eqs. (10)–(11) into a single equation with a nonlocal term. For this aim we take the Fourier transform of (11) and obtain

$$\widehat{U}(k_1, k_2) = \frac{\mathrm{i}k_1}{ck_1^2 + \gamma k_2^2}\widehat{\Lambda}(k_1, k_2)$$

where the symbol $\widehat{\ }$ denotes the Fourier transform of the associated quantity, k_1 and k_2 Fourier variables corresponding to x and y, and $\widehat{\Lambda} = \mathcal{F}(\Phi^2)$ with $\mathcal{F}$ denoting Fourier transform. Then we get $U_x = -\mathcal{K}(\Phi^2)$ where $\mathcal{K}$ is a nonlocal operator defined by

$$\mathcal{F}\{\mathcal{K}(v)\} = \alpha(k_1, k_2)\widehat{v}(k_1, k_2) \tag{19}$$

with $\alpha(k_1, k_2) = k_1^2/(ck_1^2 + \gamma k_2^2)$. If we eliminate U from (10) using the above result, we get the nonlocal equation

$$\Phi_{xx} - \tilde{\omega}\Phi + \mathcal{K}(\Phi^2)\Phi = 0. \tag{20}$$

For convenience we define a quadratic functional $\mathcal{B}$ on $L^2(\mathbb{R}^2)$ in the form:

$$\mathcal{B}(v) = \int_{\mathbb{R}^2} \mathcal{K}(v(x, y))v(x, y)\mathrm{d}x\mathrm{d}y \equiv \langle \mathcal{K}(v), v\rangle.$$

The following lemma gives Pohozaev-type identities for the nonlocal equation (20), which will be used later.

Lemma 3. *Let (ϕ, u) be a solitary wave solution of (1) and (2) in the form (9) with $\Phi \in H^1(\mathbb{R}^2)$, $\nabla U \in L^2(\mathbb{R}^2)$. Then Φ and U must satisfy*

$$-2\tilde{\omega}\int_{\mathbb{R}^2} \Phi^2\mathrm{d}x\mathrm{d}y + \mathcal{B}(\Phi^2) = 0, \tag{21}$$

$$\int_{\mathbb{R}^2} [(\Phi_x)^2 + \tilde{\omega}\Phi^2]\mathrm{d}x\mathrm{d}y - \mathcal{B}(\Phi^2) = 0. \tag{22}$$

Proof. These identities are derived by direct computations. Firstly, we multiply (20) by $x\Phi_x$ and integrate the resulting equation over $\mathbb{R}^2$. Using the Parseval theorem and the definition of $\mathcal{B}(\Phi^2)$, we obtain

$$\int_{\mathbb{R}^2} [(\Phi_x)^2 - \tilde{\omega}\Phi^2]\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^2} (\widehat{\Lambda})^2 \left(\alpha - k_1\frac{\partial\alpha}{\partial k_1}\right)\mathrm{d}k_1\mathrm{d}k_2 = 0. \tag{23}$$

Similarly, we multiply (20) by $y\Phi_y$ and integrate the resulting equation over $\mathbb{R}^2$. Using the Parseval theorem and the definition of $\mathcal{B}(\Phi^2)$, we obtain

$$\int_{\mathbb{R}^2}[(\Phi_x)^2+\tilde{\omega}\Phi^2]\mathrm{d}x\mathrm{d}y-\frac{1}{2}\int_{\mathbb{R}^2}(\widehat{\Lambda})^2\left(\alpha-k_2\frac{\partial\alpha}{\partial k_2}\right)\mathrm{d}k_1\mathrm{d}k_2=0. \tag{24}$$

Finally, if we multiply (20) by Φ and integrate the resulting equation over $\mathbb{R}^2$, after one integration by parts we obtain

$$\int_{\mathbb{R}^2}[(\Phi_x)^2+\tilde{\omega}\Phi^2]\mathrm{d}x\mathrm{d}y-\int_{\mathbb{R}^2}(\widehat{\Lambda})^2\alpha\mathrm{d}k_1\mathrm{d}k_2=0 \tag{25}$$

which is equivalent to (22). Subtracting (24) from (23), we obtain (21) where the identity

$$k_1\frac{\partial\alpha}{\partial k_1}+k_2\frac{\partial\alpha}{\partial k_2}=0$$

is used. This completes the proof. $\square$

Using the Pohozaev identities (21) and (22) we formulate a variational problem which is equivalent to (20). In this regard we show that the critical points of the variational problem defined for the functional

$$J(\Phi)=\frac{\|\Phi\|_2^2\|\Phi_x\|_2^2}{\mathcal{B}(\Phi^2)}=\frac{\|\Phi\|_2^2\|\Phi_x\|_2^2}{\langle\mathcal{K}(\Phi^2),\Phi^2\rangle} \tag{26}$$

solve (20) in the weak sense. If Φ minimizes J, then the first variation of J must be zero: $\delta J=0$. To compute δJ, we compute the first variation of $\mathcal{B}(\Phi^2)$ in the form $\delta\mathcal{B}(\Phi^2)=\langle 2\Phi\mathcal{B}'(\Phi^2),\eta\rangle$ for all $\eta\in C_c^\infty(\mathbb{R}^2)$ where $\mathcal{B}'(v)$ denotes the Frechet derivative of $\mathcal{B}(v)$. To compute $\mathcal{B}'(v)$ we first use the Parseval theorem and the linearity of the Fourier transform in the identity

$$\mathcal{B}(v+\eta)-\mathcal{B}(v)=\langle\mathcal{K}(v+\eta),v+\eta\rangle-\langle\mathcal{K}(v),v\rangle,$$

and we get

$$\mathcal{B}(v+\eta)-\mathcal{B}(v)-2\int_{\mathbb{R}^2}\mathcal{K}(v)\eta\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^2}(\widehat{\eta})^2\alpha\mathrm{d}k_1\mathrm{d}k_2.$$

Then, using $0\le\alpha(k_1,k_2)\le 1/c$ in this equation we obtain the inequality

$$\|\mathcal{B}(v+\eta)-\mathcal{B}(v)-\langle 2\mathcal{K}(v),\eta\rangle\|_2^2\le\frac{1}{c}\|\eta\|_2^2.$$

This implies that $\langle\mathcal{B}'(v),\eta\rangle=\langle 2\mathcal{K}(v),\eta\rangle$. Now we are ready to compute δJ which is given by

$$\delta J=\frac{1}{[\mathcal{B}(\Phi^2)]^2}\left\{2[\langle\Phi,\Phi\rangle\langle\Phi_x,\eta_x\rangle+\langle\Phi_x,\Phi_x\rangle\langle\Phi,\eta\rangle]\langle\mathcal{K}(\Phi^2),\Phi^2\rangle-4\langle\Phi,\Phi\rangle\langle\Phi_x,\Phi_x\rangle\langle\Phi\mathcal{K}(\Phi^2),\eta\rangle\right\}=0. \tag{27}$$

Substitution of (21) and (22) into (27) gives

$$\delta J=\frac{-4\tilde{\omega}\langle\Phi,\Phi\rangle^2}{\langle\mathcal{K}(\Phi^2),\Phi^2\rangle^2}\left\{\langle\Phi_{xx}-\tilde{\omega}\Phi+\mathcal{K}(\Phi^2)\Phi,\eta\rangle\right\}=0.$$

This implies that any nonzero critical point of J given in (26) solves (20) in the weak sense. Thus the problem of existence of solitary wave solutions reduces to the problem of existence of minimum of the nonlinear functional J.

Eq. (20) is a special case of the semi-linear elliptic equation [9,10]

$$\Delta\Phi-\tilde{\omega}\Phi+\kappa\Phi^3+b\mathcal{K}(\Phi^2)\Phi=0,\quad \Phi\in H^1(\mathbb{R}^2) \tag{28}$$

where κ and b are parameters, Δ denotes the two-dimensional Laplace operator and $\mathcal{K}(\Phi^2)$ is a nonlocal operator whose Fourier transform is in the form (19) with a general symbol α. Eq. (28) appears during the study of existence of solitary wave solutions for the Davey–Stewartson equation [9] and the generalized Davey–Stewartson equation [10]. In [9], using Lions' concentration-compactness method [11], it has been shown that there are nontrivial solutions to (28) for $\tilde{\omega}>0$, $b>0$ and $\kappa+b\alpha_m>0$ where α_m denotes an upper bound of α. In [12], the same result has been obtained using Lieb's Compactness Lemma [13]. Since our nonlocal Eq. (20) is a special case of (28) with $\kappa=0$, $b=1$, $\tilde{\omega}>0$ and $\alpha_m=1/c>0$, the above existence result is also valid for (20). This establishes the existence of the nontrivial solitary wave solutions of (1) and (2).

Theorem 4. *The LSI equations, (1) and (2), have a nontrivial solitary wave solution of the form (9) if $\gamma>0$, $c>0$ and $\omega>-c^2/4$.*

References

[1] C. Sulem, P. Sulem, The Nonlinear Schrödinger Equation Self-Focusing and Wave Collapse, Springer-Verlag, Toronto, 1999.
[2] T.D. Colin, D. Lannes, Long-wave short-wave resonance for nonlinear geometric optics, Duke Math. J. 107 (2001) 351–419.
[3] C. Babaoglu, Long-wave short-wave resonance case for a generalized Davey–Stewartson system, Chaos Solitons Fractals 38 (2008) 48–54.
[4] Y.C. Ma, L.G. Redekopp, Some solutions pertaining to the resonant interaction of long and short waves, Phys. Fluids 22 (1979) 1872–1876.
[5] A.D.D. Craik, Wave Interactions and Fluid Flows, Cambridge University Press, London, 1985.
[6] V.D. Djordjevic, L.G. Redekopp, 2-Dimensional packets of capillary-gravity waves, J. Fluid Mech. 79 (1977) 703–714.
[7] S. Erbay, Nonlinear interaction between long and short waves in a generalized elastic solid, Chaos Solitons Fractals 11 (2000) 1789–1798.
[8] P.H. Laurencot, On a nonlinear Schrodinger equation arising in the theory of water waves, Nonlinear Anal. 24 (1995) 509–527.
[9] R. Cipolatti, On the existence of standing waves for a Davey–Stewartson system, Comm. Partial Differential Equations 17 (1992) 967–988.
[10] A. Eden, S. Erbay, Standing waves for a generalized Davey–Stewartson system, J. Phys. A 39 (2006) 13435–13444.
[11] P.L. Lions, The concentration-compactness principle in the calculus of variations, The locally compact case, part 1, Ann. Inst. H. Poincaré Anal. Non Lineairé 1 (1984) 109–145.
[12] G.C. Papanicolaou, C. Sulem, P.L. Sulem, X.P. Wang, The focussing singularity of the Davey–Stewartson equations for gravity-capillarity surface waves, Phys. D 72 (1994) 61–86.
[13] E.H. Lieb, On the lowest eigenvalue of the laplacian for the intersection of two domains, Invent. Math. 74 (1984) 441–448.