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A Split-Step Fourier Method for the Complex Modified Korteweg-de Vries Equation

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Abstract—In this study, the complex modified Korteweg-de Vries (CMKdV) equation is solved numerically by three different split-step Fourier schemes. The main difference among the three schemes is in the order of the splitting approximation used to factorize the exponential operator. The space variable is discretized by means of a Fourier method for both linear and nonlinear subproblems. A fourth-order Runge-Kutta scheme is used for the time integration of the nonlinear subproblem. Classical problems concerning the motion of a single solitary wave with a constant polarization angle are used to compare the schemes in terms of the accuracy and the computational cost. Furthermore, the interaction of two solitary waves with orthogonal polarizations is investigated and particular attention is paid to the conserved quantities as an indicator of the accuracy. Numerical tests show that the split-step Fourier method provides highly accurate solutions for the CMKdV equation.

Keywords—Split-step method, Nonlinear waves, Two coupled MKdV equations, Fourier method.

1. INTRODUCTION

The complex modified Korteweg-de Vries (CMKdV) equation is a nonlinear partial differential equation given by

$$w_t + w_{xxx} + \alpha \left(|w|^2 w\right)_x = 0, \qquad -\infty < x < \infty, \quad t > 0, \tag{1}$$

where w is a complex valued function of the spatial coordinate x and the time t, α is a real parameter, and the subscripts t and x denote differentiation. This equation has been proposed as a model for the nonlinear evolution of plasma waves [1]. The same equation has also been derived to describe the propagation of transverse waves in a molecular chain model [2] and in a generalized elastic solid [3,4].

If we decompose w into its real and imaginary parts, $w = U + iV$ $(i^2 = -1)$, we obtain the coupled pair of modified Korteweg-de Vries (MKdV) equations

$$U_t + U_{xxx} + \alpha \left[\left(U^2 + V^2\right) U\right]_x = 0,$$
$$V_t + V_{xxx} + \alpha \left[\left(U^2 + V^2\right) V\right]_x = 0.$$

These two coupled nonlinear equations describe the interaction of two orthogonally polarized transverse waves. In other words, U and V represent y-polarized and z-polarized transverse

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waves, respectively, propagating in the x-direction in an xyz Cartesian coordinate system. The polarization angle θ is defined by $\tan\theta = V/U$. For some special cases, i.e., for $V = 0$ or $U = 0$ or $\theta = \theta_0 = \text{const}$ which correspond to the $0, \pi/2$ and θ_0 polarizations, respectively, these two coupled equations reduce to the single MKdV equation. In such a case, the CMKdV equation has a solution of the form

$$w(x,t) = \sqrt{\frac{2c}{\alpha}}\,\text{sech}\left[\sqrt{c}(x - x_0 - ct)\right]\exp(i\theta_0), \tag{2}$$

which represents a solitary wave initially at x_0 moving to the right with velocity c and satisfies the boundary conditions $w \to 0$ as $x \to \pm\infty$.

We now assume that w and all its derivatives converge to zero sufficiently rapidly as $x \to \pm\infty$. Solutions of the CMKdV equation subjected to these boundary conditions are known to satisfy some conservation laws [1]. According to these conservation laws, the conserved quantities

$$I_1 = \int_{-\infty}^{\infty} w\,dx, \qquad I_2 = \int_{-\infty}^{\infty} |w|^2\,dx, \qquad I_3 = \int_{-\infty}^{\infty} \left(\frac{\alpha}{2}|w|^4 - |w_x|^2\right)dx, \tag{3}$$

remain constant in time. Note that I_2 represents the theoretical L_2 norm of the system.

The CMKdV equation is known as a nonintegrable equation in the sense of the inverse scattering method. The inverse scattering method has been used to obtain some analytical solutions, i.e., soliton solutions, of nonlinear dispersive wave equations. However, its usefulness as a general tool is limited to integrable equations. Only a few analytical solutions corresponding to some special cases of the CMKdV equation are available. These analytical solutions correspond to the linearly polarized solitary waves given in equation (2) and to circularly polarized periodic waves with constant amplitudes [3]. Since no further analytical results are known, numerical studies are essential to develop an understanding of the phenomena related to the CMKdV equation.

The numerical solution of nonlinear wave equations has been the subject of many studies in recent years. Although many numerical schemes have been proposed for some well-known integrable wave equations, such as the Korteweg-de Vries (KdV) equation and the nonlinear Schrödinger (NLS) equation, there is little numerical analysis literature for nonintegrable wave equations. One of the numerical methods employed for nonlinear wave equations is the split-step method proposed by Tappert [5]. The basic idea in the split-step method is to decompose the original problem into subproblems and then to approximate the solution of the original problem by solving the subproblems sequentially. The novel aspect of the method lies in the fact that each subproblem is explicitly solvable. However, the choice of operator splitting one should use depends solely on a particular application, and generally, it is hard to tell. For nonlinear dispersive wave equations, such as the CMKdV equation that we will be solving, an appropriate approach is to decompose the original problem into linear and nonlinear subproblems which take into account purely dispersive and purely nonlinear effects, respectively. Various versions of the split-step method have been employed for the numerical solutions of the NLS equation and its generalizations [6–9], the KdV equation [9,10], and the MKdV equation [11]. In the present study, we consider the extension of the split-step method to the solution of the CMKdV equation.

We introduce a split-step Fourier method where the linear subproblem is treated exactly and the nonlinear subproblem is evaluated approximately using a fourth-order Runge-Kutta scheme. We employ a Fourier method for the spatial discretizations of both linear and nonlinear subproblems. We present three different numerical schemes which are basically the first-order, second-order, and fourth-order versions of the split-step Fourier method. In other words, the main difference among the three schemes is in the order of the splitting approximation used.

The paper is organized as follows. In Section 2, we show how the split-step Fourier method can be reformulated for the CMKdV equation. Numerical experiments with various initial conditions for the CMKdV equation are reported in Section 3. Conclusions are contained in Section 4.

Finally, we would like to point out that the equation $w_t + w_{xxx} + \alpha|w|^2 w_x = 0$, which differs in the nonlinear term, is also called the CMKdV equation [12,13].

2. THE NUMERICAL METHOD

2.1. Review of the Split-Step Method

It is best to present the split-step method as applied to a general evolution equation in the form

$$w_t = (\mathcal{L} + \mathcal{N})w, \qquad w(x,0) = w_0(x), \tag{4}$$

where $\mathcal{L}$ and $\mathcal{N}$ are linear and nonlinear operators, respectively, and $\mathcal{L}$ and $\mathcal{N}$ do not commute with each other. For instance, $\mathcal{L} = -\frac{\partial^3}{\partial x^3}$ and $\mathcal{N} = -\alpha[(|w|^2)_x + |w|^2 \frac{\partial}{\partial x}]$ for the CMKdV equation. If $\mathcal{L}$ and $\mathcal{N}$ are assumed to be t independent, a formally exact solution of equation (4) is given by

$$w(x, t + \Delta t) = \exp[\Delta t(\mathcal{L} + \mathcal{N})]w(x,t), \tag{5}$$

where Δt is the time step between the initial and final times. In general, the exponential operator in equation (5) includes an integral term. Suppose that the linear subproblem $w_t = \mathcal{L}w$, $w(x,0) = w_0(x)$ and the nonlinear subproblem $w_t = \mathcal{N}w$, $w(x,0) = w_0(x)$ have known exact solutions

$$w(x, t + \Delta t) = \exp(\Delta t\mathcal{L})w(x,t) \tag{6}$$

and

$$w(x, t + \Delta t) = \exp(\Delta t\mathcal{N})w(x,t), \tag{7}$$

respectively. The main idea in the split-step method is to approximate the exact solution of equation (4) by solving the linear and nonlinear subproblems in a given sequential order. This may be realized by replacing the exponential operator $\exp[\Delta t(\mathcal{L} + \mathcal{N})]$ in equation (5) by an appropriate combination of products of the exponential operators $\exp(\Delta t\mathcal{L})$ and $\exp(\Delta t\mathcal{N})$. This produces an error due to the noncommutativity of $\mathcal{L}$ and $\mathcal{N}$, and at this stage, the celebrated Baker-Campbell-Hausdorf (BCH) formula [14] is very useful to noticeably reduce the error. The BCH formula for two noncommutating operators A and B can be given as follows:

$$\exp(A)\exp(B) = \exp\left\{A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A - B,[A,B]] + \cdots\right\}, \tag{8}$$

where $[A,B]$ is the commutator $[A,B] = AB - BA$. According to the BCH formula, the first-order approximation of the exponential operator in equation (5) is given by

$$\varphi_1(\Delta t) = \exp(\Delta t\mathcal{L})\exp(\Delta t\mathcal{N}). \tag{9}$$

Therefore, for the first-order version of the split-step method, the advancement in time is carried out in two steps. In the first step, a so-called intermediate solution is computed by advancing the solution according to the purely nonlinear equation. In the second step, the solution is advanced according to the linear dispersive equation. In the second-order version of the method, the exponential operator in equation (5) is approximated by

$$\varphi_2(\Delta t) = \exp\left(\frac{1}{2}\Delta t\mathcal{N}\right)\exp(\Delta t\mathcal{L})\exp\left(\frac{1}{2}\Delta t\mathcal{N}\right), \tag{10}$$

which is symmetric, that is, $\varphi_2(\Delta t)\varphi_2(-\Delta t) = 1$. It is possible to construct various product formulae of arbitrary order by preserving the symmetry [15–17]. A fourth-order splitting is given in the form [17]

$$\varphi_4(\Delta t) = \varphi_2(\omega\Delta t)\varphi_2[(1 - 2\omega)\Delta t]\varphi_2(\omega\Delta t), \tag{11}$$

where $\omega = (2 + 2^{1/3} + 2^{-1/3})/3$. Note that the number of products of exponential operators increases with the order of decay of splitting error. According to Suzuki [16], the above formulae (9)–(11) have a minimal number of exponential operators for the corresponding order of splitting errors. Relations (9)–(11) are the basic schemes of the present numerical method. In general, the operators $\mathcal{L}$ and $\mathcal{N}$ in equations (9)–(11) may be interchanged without affecting the order of the method.

2.2. Spatial Discretization

Application of the numerical method requires truncation of the infinite interval to a finite interval $[a,b]$. For the numerical experiments considered, the constants a and b may be chosen sufficiently large so that the boundaries do not affect the propagation of solitary waves. We assume that $w(x,t)$ satisfies the periodic boundary condition $w(a,t) = w(b,t)$ for $(x,t) \in \mathbb{R} \times [0,T]$. If the spatial period is, for convenience, normalized to $[0,2\pi]$ using the transformation $X = 2\pi(x-a)/(b-a)$, the CMKdV equation becomes

$$w_t + \bar{\beta} w_{XXX} + \bar{\alpha} \left(|w|^2 w\right)_X = 0, \tag{12}$$

where $\bar{\alpha} = 2\pi\alpha/(b-a)$ and $\bar{\beta} = (2\pi)^3/(b-a)^3$. The interval $[0,2\pi]$ is divided into N equal subintervals with grid spacing $\Delta X = 2\pi/N$ where the integer N is even. The spatial grid points are given by $X_j = 2\pi j/N$, $j = 0,1,2,\ldots,N$. The approximate solution to $w(X_j,t)$ is denoted by $W_j(t)$. The discrete Fourier transform of the sequence $\{W_j\}$ is defined as

$$\hat{W}_k = \mathcal{F}_k\left[W_j\right] = \frac{1}{N}\sum_{j=0}^{N-1} W_j \exp\left(-ikX_j\right), \qquad -\frac{N}{2} \le k \le \frac{N}{2} - 1. \tag{13}$$

The inversion formula for the discrete Fourier transform (13) is

$$W_j = \mathcal{F}_j^{-1}\left[\hat{W}_k\right] = \sum_{k=-N/2}^{N/2-1} \hat{W}_k \exp\left(ikX_j\right), \qquad j = 0,1,2,\ldots,N-1. \tag{14}$$

Here, $\mathcal{F}$ denotes the discrete Fourier transform and $\mathcal{F}^{-1}$ its inverse. These transforms can be realized efficiently via a fast Fourier transform (FFT) algorithm. For the FFT algorithm used here, the integer N must have only prime factors 2 and 3. Derivatives of w with respect to X may also be approximated efficiently by the FFT algorithm, for example, the third-order derivative w_{XXX} at (X_j,t) is given by $\mathcal{F}_j^{-1}[-ik^3\mathcal{F}_k[W_j]]$ and so on. We approximate spatial derivatives in both linear and nonlinear subproblems using discrete Fourier transforms.

2.3. Time Integration

As already mentioned in the introduction, we consider a split-step Fourier method for the CMKdV equation. The basic idea in the split-step method is to solve the linear equation

$$w_t + \bar{\beta} w_{XXX} = 0, \tag{15}$$

and the nonlinear equation

$$w_t + \bar{\alpha}\left(|w|^2 w\right)_X = 0, \tag{16}$$

in a given sequential order. The linear problem can be solved by means of the discrete Fourier transform and the advancements in time for the linear equation (15) are performed according to

$$W_j^{m+1} = \mathcal{F}_j^{-1}\left[\exp\left(i\bar{\beta}k^3\Delta t\right)\mathcal{F}_k\left[W_j^m\right]\right]. \tag{17}$$

Here Δt is time step and W_j^m denotes the approximation to $w(X_j, m\Delta t)$. The spatial discretization of the nonlinear equation by a Fourier pseudospectral method can be written as

$$(W_j)_t + \mathcal{F}_j^{-1}\left[ik\bar{\alpha}\mathcal{F}_k\left[|W_j|^2 W_j\right]\right] = 0, \qquad j = 0,1,2,\ldots,N-1. \tag{18}$$

For the time integration of this equation, we employ a fourth-order Runge-Kutta method. Now, the total error involved in integrating from time t to time $t+\Delta t$ will be the sum of the split-step

error and the temporal discretization error of the nonlinear equation (16). In the present split-step method, we advance the solution of the CMKdV equation in various stages depending on our choice of the splitting formulae (9)–(11) at each time step, they involve the solution of linear and nonlinear subproblems. The features of the method can be listed as follows. The method is first-order, second-order, and fourth-order accurate in time depending on which splitting approximation is used. This will be verified through the numerical experiments in the next section. Note that we use a Runge-Kutta method which has the same order of accuracy as the highest-order splitting approximation considered in this paper. Furthermore, alternating the order in which steps (15) and (16) are applied from one time step to the next does not change the order of the total error. This is also verified through the numerical experiments.

We can now summarize the first-order split-step Fourier method for the CMKdV equation as follows. Given the data W_j at any time step $t = t_m$, first advance the solution according to the nonlinear part, namely, solve equation (18) using the fourth-order Runge-Kutta method for time integration. This becomes the initial data for the linear problem which is solved by Fourier transform as indicated by equation (17). The extension of the first-order split-step scheme based on equation (9) to the second-order, and fourth-order split-step schemes based on equations (10) and (11), respectively, is straightforward.

3. NUMERICAL EXPERIMENTS

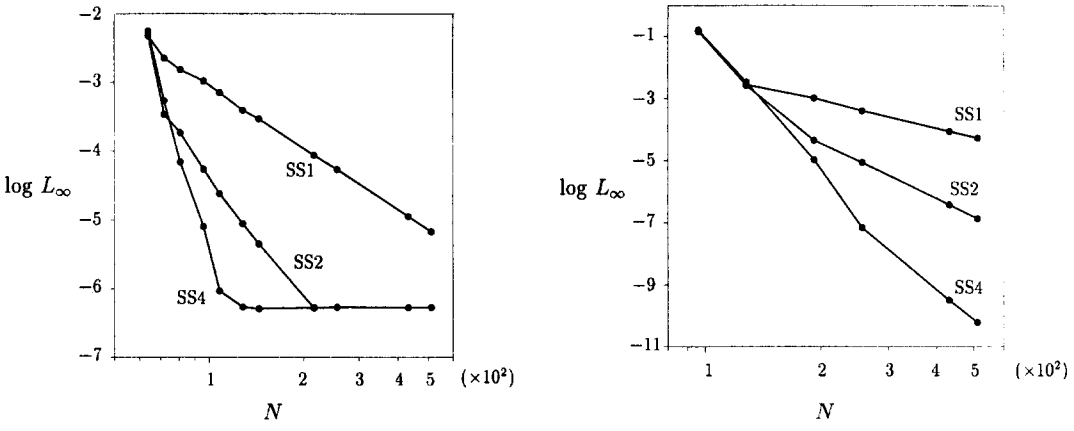
To gain insight into the performance of the suggested split-step schemes, we perform the following numerical experiments. The conservation properties of the split-step schemes are examined by calculating the conserved quantities I_2 and I_3. Note that the exact values of I_2 and I_3 for the linearly polarized solitary wave solution given by equation (2) are $I_2 = 4c^{1/2}/\alpha$ and $I_3 = 4c^{3/2}/(3\alpha)$.

3.1. Solitary Wave Solution

The purpose of the numerical experiments to be reported below is to verify numerically that the proposed split-step schemes exhibit the expected first-order, second-order, and fourth-order convergence in time and that they give spectral accuracy in space. For this reason, we restrict ourselves to problems with known analytical solutions, which correspond to the special cases of the linearly polarized single solitary wave solution given in equation (2).

We first consider a y-polarized ($\theta_0 = 0$) solitary wave solution of the CMKdV equation. The value of α is chosen as $\alpha = 2$, and the initial condition is given by $w(x, 0) = \mathrm{sech}(x - 15)$. For this problem, the exact values of the conserved quantities I_2 and I_3 are $I_2 = 2$, $I_3 = 2/3$. The problem is first solved on the interval $0 \leq x \leq 40$ for times up to $t = 10$ using the first-order, second-order, and fourth-order split-step schemes. We present in Figure 1a the L_∞-errors of the split-step schemes as a function of N for the final time $t = 10$. We use the relation $\Delta t = \nu(\Delta x)^3$ to determine the value of Δt for a given Δx, where the value of ν is fixed at $\nu = 0.025$. We note that the observed convergence rates agree well with the expected rates for the first-order, second-order, and fourth-order split-step schemes until the boundaries start exerting their influence. In particular, the nondecreasing error behavior for the second-order and fourth-order schemes after the values of $N = 216$ and $N = 128$, respectively, is due to the limited space interval $0 \leq x \leq 40$. That is, a and b must be chosen large enough so that the boundaries do not affect the solitary wave propagation. Therefore, we repeated the experiment of Figure 1a for the space interval $-20 \leq x \leq 60$. The results are presented in Figure 1b. But this time, the effect of the boundaries disappears and the L_∞-errors continue to decrease with increasing N. Then, the convergence rates agree well with the expected rates for the first-order, second-order, and fourth-order split-step schemes. The reader is referred to [18] for a discussion about the problem of finding the optimal space interval. For any fixed value of N, as might be expected, the fourth-order scheme gives smaller errors than those of the first-order and second-order schemes. Note, of course,

that the fourth-order scheme requires more computing time for a fixed number of spatial grid points. This is due to the fact that the computation of $\varphi_4(\Delta t)$ given by equation (11) requires the calculation of nine exponential operators whereas the computation $\varphi_2(\Delta t)$ (or $\varphi_1(\Delta t)$) given by equation (10) (or (9)) requires the calculation of three (or two) exponential operators. Similarly, the relation between the L_2-error and N is investigated for the split-step schemes. Since the pictures are very similar to those in Figure 1, we do not reproduce them here.



(a) The space interval $0 \leq x \leq 40$. (b) The space interval $-20 \leq x \leq 60$.

Figure 1. The L_∞-errors at $t = 10$ as a function of the number of spatial grid points for the first-order (SS1), second-order (SS2), and fourth-order (SS4) split-step Fourier schemes.

To test whether the split-step schemes exhibit the expected convergence rates in time, we perform some numerical experiments for various values of time step Δt and a fixed value of N. In these experiments, we take $N = 512$ to keep spatial accuracy high. The results are shown in Table 1. We present both the L_2-errors and the L_∞-errors for the terminating time $t = 10$. The orders of decay of the L_2-error and the L_∞-error are the ones of the splitting formulae employed for the temporal integration.

Table 1. Comparison of the convergence rates in time for the first-order, second-order, and fourth-order split-step Fourier schemes in the case of a single y-polarized solitary wave ($N = 512$, $-20 \leq x \leq 60$).

	First-Order				Second-Order				Fourth-Order			
Δt	L_∞	Order	L_2	Order	L_∞	Order	L_2	Order	L_∞	Order	L_2	Order
0.0200	2.404E-2	-	1.149E-2	-	1.217E-2	-	9.938E-3	-	9.090E-3	-	1.143E-2	-
0.0100	6.978E-3	1.785	2.714E-3	2.082	1.512E-3	3.009	6.456E-4	3.944	2.133E-4	5.413	8.577E-5	7.058
0.0020	1.183E-3	1.103	5.024E-4	1.048	6.070E-5	1.998	2.564E-5	2.004	3.630E-7	3.962	1.538E-7	3.929
0.0010	5.790E-4	1.031	2.495E-4	1.010	1.517E-5	2.000	6.410E-6	2.000	2.287E-8	3.988	9.731E-9	3.982
0.0004	2.285E-4	1.015	9.946E-5	1.004	2.427E-6	2.000	1.025E-6	2.001	5.857E-10	4.000	2.504E-10	3.994
0.0002	1.137E-4	1.007	4.967E-5	1.002	6.069E-7	2.000	2.564E-7	1.999	4.267E-11	3.779	1.881E-11	3.735

To test whether the split-step schemes exhibit the expected convergence rate in space, we perform some further numerical experiments for various values of N and a fixed value of the time step Δt. In these experiments, we take $\Delta t = 0.95366 \times 10^{-4}$ to minimize temporal errors. Furthermore, we make the calculations about the rate of convergence under the assumption that the method is an algebraically convergent method in space. In other words, we suppose that $E(N) \sim N^{-p}$ for some $p > 0$ where $E(N)$ is the L_2 (or L_∞) error when using N subintervals. The results corresponding to the fourth-order split-step scheme are shown in Table 2 for an increasing number of subintervals. We present the L_2-errors and the L_∞-errors for the terminating time $t = 10$ together with the observed rates of convergence in each case. These results show that

our fourth-order split-step scheme converges rapidly to the accurate solution in space, which is indicative of exponential convergence. Since the results corresponding to the first-order and second-order schemes are almost identical to those given in Table 2, we do not reproduce them here.

Table 2. The convergence rate in space for the fourth-order split-step Fourier scheme in the case of a single y-polarized solitary wave ($\Delta t = 0.95366 \times 10^{-4}$, $-20 \leq x \leq 60$).

N	L_∞	Order	L_2	Order
64	3.134E-0	-	2.934E-0	-
96	1.490E-1	7.51	6.922E-2	9.24
128	3.181E-3	13.37	2.815E-3	11.13
144	7.077E-4	12.76	7.693E-4	11.01
192	1.051E-5	14.63	9.144E-6	15.41
216	9.272E-7	20.61	5.280E-7	24.21

To compare the proposed split-step Fourier schemes in terms of computational efficiency, we fix Δt and Δx and measure the computing times, the L_2-error, the L_∞-error, and the conservation errors δ_2 and δ_3 at the terminating time $t = 10$. Here, δ_2 and δ_3 represent the relative errors in discrete approximations to the conservation integrals in I_2 and I_3, respectively. They are defined by $\delta_2 = |\bar{I}_2 - \bar{I}_{20}|/|\bar{I}_{20}|$ and $\delta_3 = |\bar{I}_3 - \bar{I}_{30}|/|\bar{I}_{30}|$, where $\bar{I}_2, \bar{I}_3$ and $\bar{I}_{20}, \bar{I}_{30}$ represent the calculated values of the conserved quantities I_2, I_3 at times $t = 10$ and $t = 0$, respectively. Simpson's rule is used for the numerical quadrature of the integrals. The results are represented in Table 3. Note that the computing times in Table 3 are normalized so that the computing time of the first-order split-step scheme is one unit. The results show that each of the conserved quantities is very well preserved for the split-step schemes. Furthermore, as might be expected, the fourth-order scheme is more accurate but more expensive than the others. Note that a replacement of the second-order scheme by the fourth-order scheme reduces the error by a factor of 1000. However, the computational cost is multiplied only by 2.5. Similar observations are also valid for a replacement between the first-order and second-order schemes. We conclude that the fourth-order split-step scheme is a good method for situations where accuracy rather than computational cost is important.

Table 3. Comparison of the L_∞-error, the L_2-error, the relative errors δ_2 and δ_3 in the conserved quantities, and the computing times for the first-order, second-order, and fourth-order split-step Fourier schemes in the case of a single y-polarized solitary wave ($N = 512$, $-20 \leq x \leq 60$, $\Delta t = 0.95366 \times 10^{-4}$).

Method	L_∞	L_2	δ_2	δ_3	Normalized CPU
First-Order	5.411E-05	2.367E-05	5.414E-12	4.638E-08	1.0
Second-Order	1.380E-07	5.829E-08	5.395E-12	1.719E-11	1.6
Fourth-Order	6.081E-11	2.681E-11	1.349E-11	4.027E-11	4.1

The same experiments are also carried out for a z-polarized ($\theta_0 = \pi/2$) solitary wave solution of the CMKdV equation. We have found that the results obtained for a z-polarized solitary wave are identical to those given for a y-polarized solitary wave above. This is natural because for these special cases, the CMKdV equation reduces to the single MKdV equation and the above single solitary wave solutions are, in fact, the one-soliton solutions of the MKdV equation.

3.2. Interacting Solitary Waves

Here, we first study the interaction of two orthogonally polarized solitary waves, namely, the interaction between a y-polarized ($\theta_0 = 0$) solitary wave and a z-polarized ($\theta_0 = \pi/2$) solitary wave. The value of α is chosen as $\alpha = 2$, and the initial condition is given by

$$w(x, 0) = w_1(x, 0) + iw_2(x, 0),$$

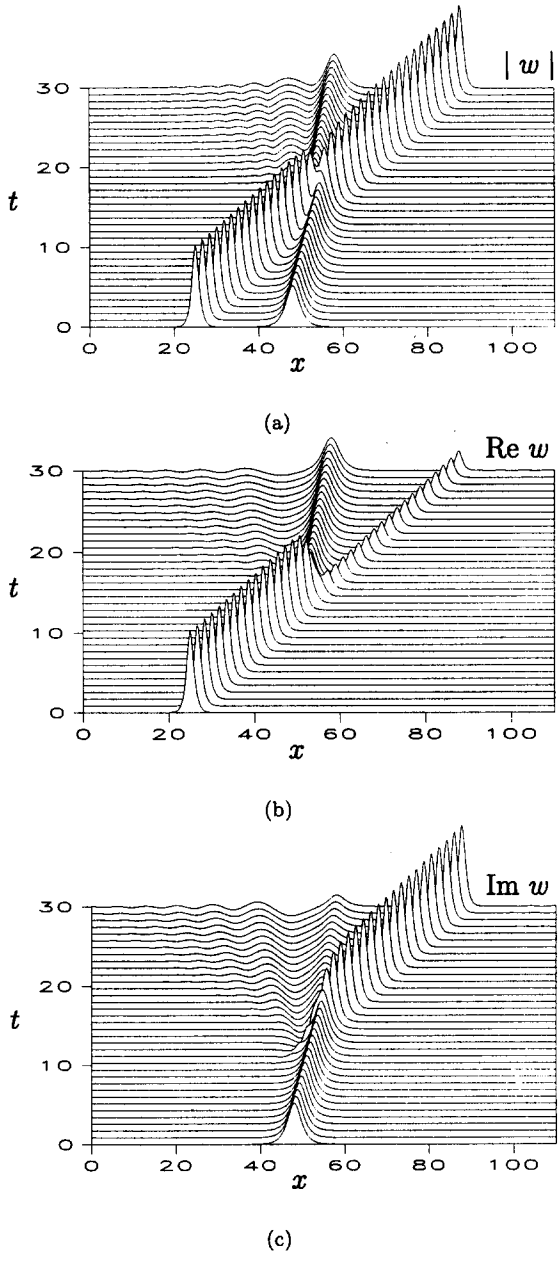


Figure 2. The modulus (a), real (b), and imaginary (c) parts of the numerical solution obtained for the interaction between a y-polarized and a z-polarized solitary waves using the fourth-order split-step Fourier scheme with $N = 1024$ and $\Delta t = 0.3099 \times 10^{-4}$.

where $w_1(x,0)$ and $w_2(x,0)$ are defined as

$$w_1(x,0) = \sqrt{2}\,\mathrm{sech}\left[\sqrt{2}(x-25)\right], \qquad w_2(x,0) = \frac{1}{\sqrt{2}}\,\mathrm{sech}\left[\frac{1}{\sqrt{2}}(x-48)\right]. \tag{19}$$

These equations correspond to two solitary waves, the one initially located at $x = 25$ and moving to the right with speed 2, the one initially located at $x = 48$ and moving to the right with speed $1/2$. For this problem, the exact values of the conserved quantities are $I_2 = 3\sqrt{2}$, $I_3 = 3/\sqrt{2}$. The problem is solved on the interval $0 \le x \le 110$ for times up to $t = 30$ using the proposed split-step schemes.

Figure 2 shows the modulus and the real and imaginary parts of the numerical solution obtained using the fourth-order scheme. These results are obtained using $N = 1024$ spatial grid points

and a time step of $\Delta t = 0.3099 \times 10^{-4}$. We note that initially the taller one is a y-polarized wave, whereas the shorter one is a z-polarized wave. Since the taller one moves faster than the shorter one, it catches up and collides with the shorter one, then moves away from the shorter one as time increases. It is surprising that the polarizations of the solitary waves are approximately interchanged after the collision. In other words, after the collision, the taller one is approximately a z-polarized wave, whereas the shorter one is a y-polarized wave. The nonexistence of purely y-polarized and z-polarized waves shows that energy is exchanged during the interaction. We also note that there is a tail following the shorter one after the interaction. This is what is reported in [19] for a similar experiment. In this respect, the above solitary wave solutions are not solitons which do not exchange energy during collision. These results are in agreement with the fact that the CMKdV equation cannot be solved using the inverse scattering transform as it is nonintegrable in the terminology of nonlinear wave theory. The plots in Figure 2 also show that the y-polarized wave has moved forward and the z-polarized one backward, relative to the positions they would have reached if the interaction were linear.

Since an analytical solution is not available for this problem, we cannot present the L_2- and L_∞-errors. But, as a numerical check on the accuracy of the fourth-order split-step scheme, we present in Figures 3a and 3b the evolution of the numerical errors in the conserved quantities I_2 and I_3, respectively, for the above experiment. These results are again obtained using $N = 1024$ spatial grid points and a time step of $\Delta t = 0.3099 \times 10^{-4}$. As can be seen from Figure 3, both quantities remain constant with respect to time t and this behavior provides a valuable check on the numerical results. Notice how the results in Figure 3 differ from those in Table 3, showing that this problem is more difficult to treat numerically than those treated above. We find that further spatial resolution is needed to capture the dynamics related to the interaction of two solitary waves.

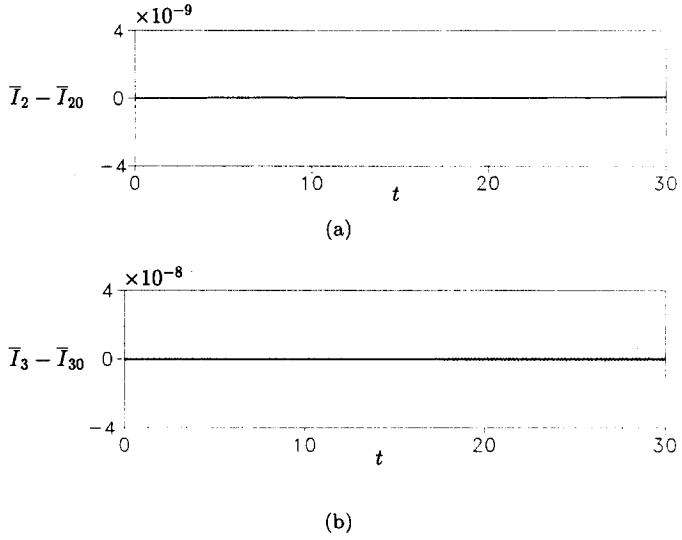


Figure 3. The numerical errors in the conserved quantities I_2 (a) and I_3 (b) as functions of time for the interaction between a y-polarized and a z-polarized solitary waves using the fourth-order split-step Fourier scheme with $N = 1024$ and $\Delta t = 0.3099 \times 10^{-4}$.

Finally, we study the interaction between two y-polarized ($\theta_0 = 0$) solitary waves. The value of α is chosen as $\alpha = 2$, and the initial condition is given by $w(x,0) = w_1(x,0) + w_2(x,0)$, where $w_1(x,0)$ and $w_2(x,0)$ are given by equation (19) again. The exact values of the conserved quantities are again $I_2 = 3\sqrt{2}$, $I_3 = 3/\sqrt{2}$. The problem is solved on the interval $0 \leq x \leq 110$ for times up to $t = 30$ using the fourth-order split-step scheme.

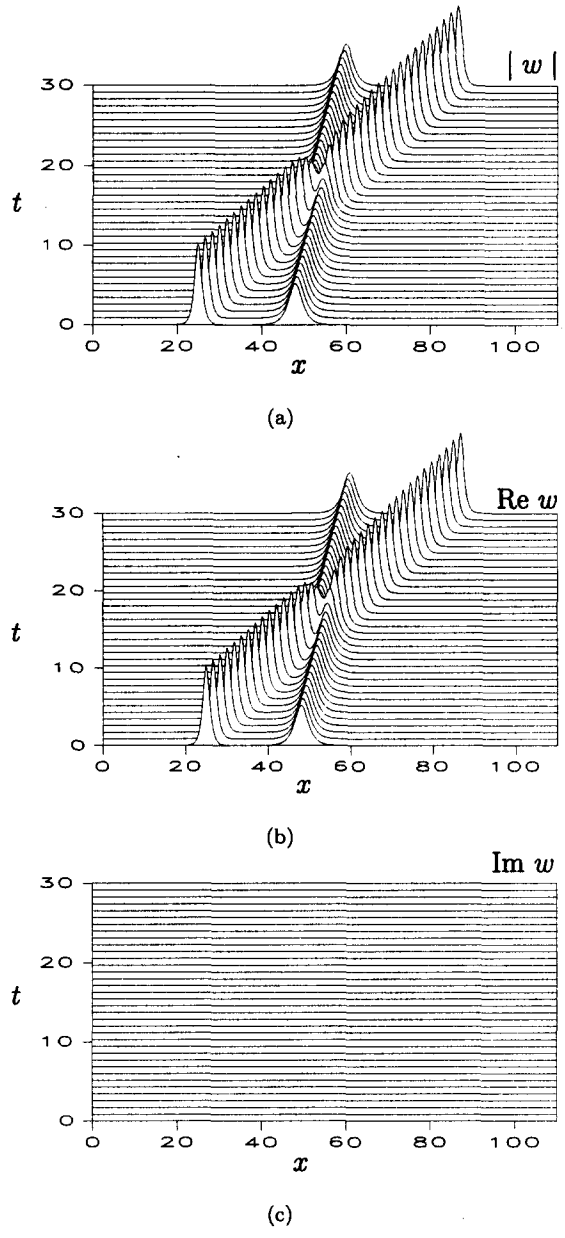


Figure 4. The modulus (a), real (b), and imaginary (c) parts of the numerical solution obtained for the interaction between two y-polarized solitary waves using the fourth-order split-step Fourier scheme with $N = 768$ and $\Delta t = 0.7345 \times 10^{-4}$.

Figure 4 shows the evolution of the modulus and the real and imaginary parts of the numerical solution obtained using the fourth-order scheme. These results are obtained using $N = 768$ spatial grid points and a time step of $\Delta t = 0.7345 \times 10^{-4}$. We note that initially both the taller one and the shorter one are purely y-polarized solitary waves. It is surprising that the polarizations of the solitary waves remain the same after the collision. In other words, both the taller one and the shorter one are still purely y-polarized solitary waves. This shows that the solitary waves are stable under collisions when the two solitary waves are polarized in the same direction. The taller one weakens during the interaction and then recovers. The observation that the interaction is clean is in agreement with the fact that the CMKdV equation reduces to the single MKdV equation for this special case. That is, the above y-polarized solitary waves are, in fact, the two-soliton solutions of the single MKdV equation. As a further check on this

conclusion, we investigate the evolution of the numerical errors in the conserved quantities I_2 and I_3, respectively, for the present experiment. As is expected, both quantities remain constant with respect to time t. Since the pictures are very similar to those in Figure 3, we do not reproduce them here.

4. CONCLUSIONS

In this study, we have extended the well-known split-step Fourier method for solving some nonlinear wave equations to the CMKdV equation. We have presented three split-step schemes which are characterized by the operator splitting approximation used. The numerical solutions obtained by using the present numerical schemes for the case of one solitary wave are compared with the exact solutions in order to assess the accuracy of these schemes. In addition, the performance of the numerical schemes has been monitored by computing both the conserved quantities and the computational costs. We have found that the numerical results are in good agreement with the exact solutions and the results reported in the literature. Moreover, the collision of two solitary waves with orthogonal polarizations, whose behavior is still analytically unknown, is investigated numerically and it is observed that the polarizations of the solitary waves are approximately interchanged after the collision.

The numerical experiments reported here show that the fourth-order split-step Fourier scheme is advisable in situations where accuracy rather than the computational cost is of prime importance. Naturally, the algorithm for the fourth-order scheme is more expensive than the ones for the first-order and the second-order schemes: the difference lies in the fact that the computation of $\varphi_4(\Delta t)$ given by the splitting formula (11) requires the calculation of nine exponential operators, whereas the computations of $\varphi_1(\Delta t)$ and $\varphi_2(\Delta t)$ given by the splitting formulae (9) and (10), respectively, require the calculation of two and three exponential operators only. It should be stressed that a higher-order splitting approximation is introduced to increase the accuracy and that this is done at the cost of increasing computational time. A proper choice of the splitting approximation is a compromise between these competing effects.

We want to point out that an interchange of the operators $\mathcal{L}$ and $\mathcal{N}$ in equations (9)–(11) does not affect the orders of the schemes. This has been confirmed in the numerical experiments considered.

The question naturally arises as to what extent the results obtained for a solitary wave with constant polarization angle are applicable to more general problems for which the analytical solutions are not available. At this point, we believe that the remarkable properties of the numerical schemes related to the conserved quantities provide a valuable check on the numerical results. However, we think that the present split-step Fourier method clearly deserves further study with respect to its application to the CMKdV equation, which is, in general, a nonintegrable evolution equation in the sense of the inverse scattering transform.

There are still very interesting questions concerning the interaction of solitary waves for the CMKdV equation. For instance, it is interesting to ask how a nonlinear interaction of two solitary waves, in which the initial polarization difference is an acute angle, possibly changes their initial polarizations. It would also be interesting to examine whether a collision initiates the process of exciting solutions with polarization different than the original. Such problems are currently under investigation and the method proposed in this report is believed to provide insight into the computation of the polarization dynamics phenomena associated with the CMKdV equation.

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