PERGAMON

International Journal of Engineering Science 38 (2000) 1497–1516

International Journal of Engineering Science

www.elsevier.com/locate/ijengsci

An asymptotic theory of thin micropolar plates

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Received 26 October 1998; received in revised form 14 June 1999; accepted 26 July 1999

Abstract

The asymptotic expansion technique is used to obtain the two-dimensional dynamic equations of thin micropolar elastic plates from the three-dimensional dynamic equations of micropolar elasticity theory. To this end, all the field variables are scaled via an appropriate thickness parameter such that it reflects the expected behavior of the plate. A formal power series expansion of the three-dimensional solution is used by considering the thickness parameter as a small parameter. Without any a priori assumption on the form of the field variables, it is shown that the zeroth-order approximation simultaneously includes both the plate equations previously presented in the literature and the standard assumptions on the specific forms of the field variables. Some aspects of the present asymptotic plate equations are also discussed.

Keywords: Micropolar; Plates; Asymptotic

1. Introduction

The purpose of the present work is to derive an asymptotic theory of thin micropolar plates in the framework of the theory of linear micropolar elasticity and, then, is to show that the present derivation completely agrees with those obtained previously in the literature.

The classical plate theories including the celebrated Kirchhoff theory which is the most popular linear model of thin elastic plates are usually obtained under a variety of ad hoc assumptions. These two-dimensional models which are, in fact, two-dimensional approximations of three-dimensional elastic plates involve a priori assumptions regarding the variation of the unknowns (i.e., the displacements and the stresses) across the thickness of the plate (for instance the

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PII: S0020-7225(99)00118-4

celebrated Love–Kirchhoff asssumption). The assumptions on which the theory of small deflections of thin elastic plates is based may be found for example in [1]. Another method which has been used to obtain two-dimensional models of thin elastic plates is the so-called asymptotic expansion method. In this method, a formal power series expansion of the three-dimensional solution is used by considering the thickness of the plate as the small parameter and the Kirchhoff model of linear elastic isotropic plates is obtained as the leading term of the formal asymptotic expansion. The early works of Goldenveizer [2] and Friedrichs and Dressler [3], among many others, are the representative examples of this approach. An historical account of the subject, together with references to other and related contributions on the subject, may be found in the monograph by Naghdi [4]. A rigorous mathematical reformulation of the asymptotic approach has been given by Ciarlet and Destuynder [5] in which the three-dimensional problem is posed in variational form and a functional framework is used. Later on, Ciarlet [6] has derived the von Kármán plate equations corresponding to geometrically nonlinear, but materially linear, equations of elasticity. The asymptotic approach has also been applied by some authors to derive the fully nonlinear models of thin plates that are made of a general nonlinear elastic material. An overview of this work along with an asymptotic membrane theory of thin hyperelastic plates is given by Erbay [7].

Most of the early derivations of plate models via asymptotic analysis are typically restricted to the classical theory of elasticity. However, various theories have been proposed in recent years to incorporate the internal, discrete, structure of matter into the classical elasticity model. These theories take different names depending on which aspect of continuum has been chosen as a starting point. Higher order gradients, non-local particle interactions, polyatomic structure and local intrinsic rotations are some of these aspects, among others. The question naturally arises as to whether asymptotic approach generalize to such theories of continua with internal structures. One objective of this work is to show that such an extension of the asymptotic expansion method is possible for one of the above-mentioned generalized continuum theories, i.e., micropolar elasticity theory.

The theory of micropolar elasticity is concerned with an elastic medium whose constituents, the so-called material points, are allowed to rotate independently without stretch. In other words, a single vector assigned to every point of the medium may be considered as the additional kinematical ingredient in the theory. Hence, the motion of the material points of such a medium will have three additional degrees of freedom associated with local micro-rigid rotations. The fundamental equations of a micropolar elastic medium then contain coupled microrotation and displacement fields [8]. A first attempt to construct a plate theory based on the linear theory of micropolar elasticity was made by Eringen [9]. The approach of Eringen to the derivation of a micropolar plate theory can be described as follows. Assuming that the stress and displacement fields do not vary violently across the thickness of the plate, the three-dimensional field equations of micropolar elasticity are integrated across the thickness and the balance equations of micropolar elastic plates are derived. Next, as the Love–Kirchhoff assumption in the derivation of classical plate theory, a priori assumptions regarding the variation of the unknowns across the thickness of the plate are made below:

(i) The transverse displacement component is independent of the transverse coordinate. The in-plane displacement components are not only functions of the in-plane coordinates but they are also linear functions of the transverse coordinate.

(ii) Both transverse and in-plane components of the microrotation vector are independent of the transverse coordinate.

Formulations are made for the case of either vanishing transverse normal strain or transverse normal stress. As a consequence, the governing equations which describe both the extensional motions and the flexural motions of a micropolar elastic plate are obtained. Then various approximate micropolar plate models are derived by neglecting some terms in the micropolar plate equations. Later on, Green and Naghdi [10] derived a different set of micropolar plate equations by using a method of asymptotic expansion. Furthermore, they discussed the relationship between micropolar and director theories of plates and the relations of these to a theory derived for a Cosserat plate. They concluded that the method used by Eringen does not provide a consistent approximate theory of plates based on three-dimensional equations of micropolar elasticity. In particular, they claimed that the micropolar plate theory of Eringen appears to be a first approximation together with some, but not all, terms of a second approximation. In this paper we show that the micropolar plate equations of Eringen can be derived from the three-dimensional dynamical equations of linear micropolar elasticity by considering asymptotic expansions in terms of the thickness of the plate. In other words, we conclude that two different micropolar plate theories presented in [9,10] correspond to the leading orders of two different asymptotic expansions. The basic reason for the difference between the theories presented in [9,10] seems to be the fact that they are based on the different assumptions related to the variations of the in-plane displacement components across the thickness. In other words, in the theory of Green and Naghdi the in-plane displacement components are independent of the transverse coordinate for a first approximation, while in the theory of Eringen they are linear functions of the transverse coordinate. For additional references about various aspects of the micropolar plate equations of Eringen we refer the reader to Wang [11] and references therein.

We remark that the present analysis does not discuss edge boundary conditions and that the asymptotic method usually gives rise to a theory which is not valid near the edges of the plate. The classical Kirchhoff model of linear elastic isotropic plates, which is obtained as the leading term of the formal asymptotic expansion of the three-dimensional solution, requires a couple of boundary conditions at a cylindrical edge of the plate. This is consistent with the fourth-order differential equation of the classical plate theory. However, from the standpoint of three-dimensional elasticity theory, three independent boundary conditions must be satisfied along the edge of the plate. In [3], Friedrichs and Dressler showed by the asymptotic expansion method how the three boundary conditions degenerate into the two boundary conditions. Then, the exact solution of the plate problem consists of an outer solution which is valid away from the plate edge and a boundary layer solution which is only significant in a narrow region adjacent to the edge of the plate. The question how the boundary-layer analysis of the classical plate theory can be extended to the micropolar elasticity theory is an interesting question. However, it is not the purpose of the present study to analyze the boundary layer behavior of the micropolar elastic plates. Therefore, the analysis to be presented here should be considered as an outer solution of the actual boundary value problem and the boundary-layer effects are outside the scope of the present paper. We also remark that we apply the asymptotic expansion method directly to the field equations in local form, which is different than that of the work of Ciarlet where a variational formulation is used.

In Section 2, the notation is established and the three-dimensional field equations of the linearized theory of micropolar elasticity are stated. Later, utilizing an appropriate thickness parameter, all the field variables are scaled such that it reflects the expected behavior of the plate. Then, the three-dimensional problem is transformed into a problem over a domain whose dimensions are of comparable order.

In Section 3, the displacement vector, the microrotation vector, the stress tensor and the couple stress tensor are expanded in powers of the thickness parameter and the hierarchies of the field equations are obtained. The equations corresponding to the leading order approximation are studied in detail. It is shown that the leading order displacement field is a Kirchhoff–Love field, and the in-plane components of the leading order microrotation vector are independent of the transverse coordinate. Moreover, the stress and couple-stress components in the transverse direction are calculated in terms of the leading order displacement and microrotation fields.

In Section 4, various aspects of the present theory are discussed. It is shown that the resulting equations of the present theory reduce to the plate equations of Eringen as a special case. Finally, the speeds of flexural waves propagating in an infinite micropolar elastic plate are calculated and compared with those obtained previously.

2. The three-dimensional problem and scaling

In this section we first establish the notation and describe the three-dimensional dynamical problem for a plate made of a homogeneous isotropic linear micropolar elastic material. Next we scale the unknowns and loads by some particular powers of the thickness parameter and express the problem in the dimensionless coordinates.

2.1. Notation and three-dimensional equations

We consider a micropolar elastic plate of constant thickness $2h$. The material particles of the plate are identified by their Cartesian coordinates x_k $(k = 1, 2, 3)$ in its reference configuration. It is assumed that the middle surface of the unstressed plate lies in the x_1x_2 plane. The edge of the middle surface of the plate is bounded by a smooth curve Γ lying in the x_1x_2 plane. The upper and lower surfaces of the plate are defined by $x_3 = h$ and $x_3 = -h$, respectively.

Before proceeding, the following conventions are adopted. Throughout the text, Greek indices take values in the set $\{1, 2\}$, whereas Latin indices take values in $\{1,\ 2,\ 3\}$ unless stated otherwise, and repeated indices indicate that the summation convention is being used. Also the indices following a comma denote the partial differentiation with respect to spatial coordinates and a superposed dot indicates the partial differentiation with respect to time.

The component form of the balance equations of micropolar elasticity, which are decomposed into the in-plane and transverse components, are given by (for details, see [8])

Balance of linear momentum

$$t_{\alpha\beta,\alpha} + t_{3\beta,3} + \rho f_\beta = \rho \ddot{u}_\beta, \qquad t_{\alpha 3,\alpha} + t_{33,3} + \rho f_3 = \rho \ddot{u}_3. \tag{2.1}$$

Balance of angular momentum

$$m_{\alpha\beta,\alpha} + m_{3\beta,3} + \epsilon_{\alpha\beta}(t_{3\alpha} - t_{\alpha 3}) + \rho l_\beta = \rho j \ddot{\varphi}_\beta, \qquad m_{\alpha 3,\alpha} + m_{33,3} + \epsilon_{\alpha\beta} t_{\alpha\beta} + \rho l_3 = \rho j \ddot{\varphi}_3, \tag{2.2}$$

where $\epsilon_{\alpha\beta}$ is the two-dimensional permutation symbol defined by

$$\epsilon_{11} = \epsilon_{22} = 0, \quad \epsilon_{12} = -\epsilon_{21} = 1.$$

In these equations, ρ is the density, j the inertia density, f_k the body force, l_k the body couple, u_k the displacement vector, φ_k the microrotation vector, t_{kl} the stress tensor and m_{kl} is the couple stress tensor.

We assume that the following boundary conditions hold on the upper and lower surfaces of the plate:

$$\left.\begin{array}{ll} t_{3\beta} = \mp t_\beta^\mp, & t_{33} = \mp t_3^\mp \\ m_{3\beta} = \mp m_\beta^\mp, & m_{33} = \mp m_3^\mp \end{array}\right\} \quad \text{at } x_3 = \mp h, \tag{2.3}$$

where $t_k^\mp$ and $m_k^\mp$ are the surface tractions and the surface couples, respectively, prescribed on the upper and lower surfaces of the plate.

The linear theory of micropolar elasticity include two sets of constitutive equations, one for the stress and one for the couple stress. Thus, the explicit forms of the stress tensor, t_{kl}, and the couple stress tensor, m_{kl}, are given as follows (for details, see [8]):

Stress components

$$\begin{aligned} t_{\alpha\beta} &= \lambda(u_{\gamma,\gamma} + u_{3,3})\delta_{\alpha\beta} + \mu u_{\alpha,\beta} + (\mu+\kappa)u_{\beta,\alpha} - \kappa\epsilon_{\alpha\beta}\varphi_3, \\ t_{\alpha 3} &= \mu u_{\alpha,3} + (\mu+\kappa)u_{3,\alpha} + \kappa\epsilon_{\alpha\beta}\varphi_\beta, \\ t_{3\alpha} &= \mu u_{3,\alpha} + (\mu+\kappa)u_{\alpha,3} - \kappa\epsilon_{\alpha\beta}\varphi_\beta, \\ t_{33} &= \lambda u_{\gamma,\gamma} + (\lambda + 2\mu + \kappa)u_{3,3}. \end{aligned} \tag{2.4}$$

Couple stress components

$$\begin{aligned} & m_{\alpha\beta} = \alpha(\varphi_{\gamma,\gamma} + \varphi_{3,3})\delta_{\alpha\beta} + \beta\varphi_{\alpha,\beta} + \gamma\varphi_{\beta,\alpha}, \quad m_{\alpha 3} = \beta\varphi_{\alpha,3} + \gamma\varphi_{3,\alpha}, \\ & m_{3\alpha} = \beta\varphi_{3,\alpha} + \gamma\varphi_{\alpha,3}, \quad m_{33} = \alpha\varphi_{\gamma,\gamma} + (\alpha+\beta+\gamma)\varphi_{3,3}, \end{aligned} \tag{2.5}$$

where λ, μ, κ, α, β and γ are linear material constants.

2.2. Scaling

The thickness parameter ε is defined in the form $\varepsilon = h/L$ where $2L$ is a characteristic lateral length scale. The thin plate assumption results in $\varepsilon \ll 1$. Consequently, we may attempt to study an asymptotic expansion of the basic equations given above with respect to the small dimensionless parameter ε. We now introduce the following dimensionless coordinates $\bar{x}_k$ $(k = 1, 2, 3)$:

$$x_\alpha = L\bar{x}_\alpha, \quad x_3 = \varepsilon L\bar{x}_3, \quad \bar{x}_3 \in [-1, 1] \tag{2.6}$$

and the dimensionless time $\bar{t}$

$$t = \left(\frac{\rho}{\mu}\right)^{1/2} L\varepsilon^{-1}\bar{t}. \tag{2.7}$$

Thus, the domain occupied by the thin plate is transformed into a domain of comparable dimensions, which is independent of ε. Depending on the scale of time which characterizes the variability of the state of stress in time, Gusein-Zade [12] proposed a classification of the dynamic processes occurring in a thin plate. According to this classification, our scaling of time in which the value of the exponent of ε is -1 corresponds to the case where the stretching problem is of quasistatic character and the bending problem reduces to the dynamic equations of the classical plate theory.

Since such a transformation introduces the thickness parameter ε into the field equations, the solution of the three-dimensional problem defined in Eqs. (2.1)–(2.5) will depend not only on $\bar{\boldsymbol{x}}$ but also on ε. It is reasonable then to expect the solution of the corresponding problem defined in the dimensionless coordinates to have an expansion with respect to ε. In what follows, all the field variables in dimensionless coordinates will be scaled by some particular powers of ε such that it reflects the expected behavior of the plate and then a consistent asymptotic plate theory for thin micropolar elastic plates will be developed without any additional assumptions. But at this point, note that the asymptotic expansion method is not free from a priori assumptions due to the scaling assertions. Now, remembering that the plate thickness is much smaller than the typical length of the plate, we define the non-dimensional displacement vector and stress tensor components in $\bar{\boldsymbol{x}}$-coordinates, i.e., $\bar{u}_k$ and $\bar{t}_{kl}$, as follows:

$$\begin{aligned} &u_\alpha = \varepsilon^{2+a} L \bar{u}_\alpha, \quad u_3 = \varepsilon^{1+a} L \bar{u}_3, \qquad a > 0, \\ &t_{\alpha\beta} = \mu\varepsilon^{2+a}\bar{t}_{\alpha\beta}, \quad (t_{\alpha 3}, t_{3\alpha}) = \mu\varepsilon^{3+a}(\bar{t}_{\alpha 3}, \bar{t}_{3\alpha}), \quad t_{33} = \mu\varepsilon^{4+a}\bar{t}_{33}, \end{aligned} \tag{2.8}$$

where a is a fixed positive real number and the common factor of stress dimension is chosen as μ. As is seen from (2.8), the transverse displacement component u_3 is assumed to be small compared to the plate thickness. On the other hand, the scale of the lateral displacement components u_α is chosen such that the lateral displacements are much smaller than the transverse displacement. Moreover, the scaling of planar components of the strain tensor implies the scaling of in-plane components of the stress tensor. The scales of the transverse components of the stress tensor are chosen such that the transverse shear stresses $t_{\alpha 3}$ and $t_{3\alpha}$ and the transverse normal stress t_{33} are small compared with the in-plane stresses $t_{\alpha\beta}$. From the point of view of the theory of small deflections of plates the present approach seems physically reasonable. Finally, note that the scaling in (2.8) agrees with those proposed in [5,13] to derive the linear plate theory.

To proceed with our analysis further, we introduce the scaling

$$\begin{aligned} &\varphi_\alpha = \varepsilon^{1+a}\overline{\varphi}_\alpha, \quad \varphi_3 = \varepsilon^{2+a}\overline{\varphi}_3, \\ &m_{\alpha\beta} = \mu L\varepsilon^{3+a}\overline{m}_{\alpha\beta}, \quad (m_{\alpha 3}, m_{3\alpha}) = \mu L\varepsilon^{4+a}(\overline{m}_{\alpha 3}, \overline{m}_{3\alpha}), \quad m_{33} = \mu L\varepsilon^{5+a}\overline{m}_{33} \end{aligned} \tag{2.9}$$

for the microrotation vector and couple stress tensor components. This particular choice of scaling for the microrotation vector components can be justified as follows: Suppose all the displacement and stress components are independent of x_3 for the time being. Then, it is easily seen that Eqs. (2.1) and (2.2) couple the microrotation φ_3 with the in-plane displacements u_α whereas the microrotations φ_α with the transverse deflection u_3. Recalling that the lateral displacements are assumed to be small compared to the transverse displacement, we also expect that the microrotation φ_3 is smaller than the microrotations φ_α. After these observations, φ_α and φ_3

are assumed to be of the same order of magnitude as u_3 and u_α, respectively. In order to non-dimensionalize the couple stress components we need a common factor of length dimension beside the unit of stress, μ. We have to choose between two options: either the typical length of the plate, L, or the plate thickness h. Recalling that the theory of micropolar elasticity has been suggested to model physical phenomena in which the length scale is comparable to distances and sizes of the microelements, we choose the latter, i.e., the plate thickness, which is much smaller than the typical length of the plate. Thus, the couple stress tensor components are shifted to the next order compared with the stress tensor components in the present theory.

We define the non-dimensional surface and body forces in the following form:

$$\begin{aligned} t_\alpha^\mp &= \mu\varepsilon^{3+a}\overline{t}_\alpha^\mp, \quad t_3^\mp = \mu\varepsilon^{4+a}\overline{t}_3^\mp, \\ \rho L f_\alpha &= \mu\varepsilon^{2+a}\overline{f}_\alpha, \quad \rho L f_3 = \mu\varepsilon^{3+a}\overline{f}_3. \end{aligned} \tag{2.10}$$

Similarly, the non-dimensional surface and body couples are defined by

$$\begin{aligned} m_\alpha^\mp &= \mu L\varepsilon^{4+a}\overline{m}_\alpha^\mp, \quad m_3^\mp = \mu L\varepsilon^{5+a}\overline{m}_3^\mp, \\ \rho l_\alpha &= \mu\varepsilon^{3+a}\overline{l}_\alpha, \quad \rho l_3 = \mu\varepsilon^{4+a}\overline{l}_3. \end{aligned} \tag{2.11}$$

The above quantities introduced in Eqs. (2.10) and (2.11) are scaled such that the non-dimensional forms of these quantities are included in the lowest order approximation and external forces and couples tend to zero with the vanishing thickness. Again, since we choose the plate thickness as the appropriate length scale for the surface and body couples, the surface and body couples are shifted to the next order compared with the surface and body forces.

Recalling that μ has been chosen as the unit of stress whereas h has been chosen as the unit of length for the micropolar effects, the non-dimensional material constants are introduced in the following form:

$$\lambda = \mu\overline{\lambda}, \quad (\alpha, \beta, \gamma) = \mu L^2\varepsilon^2(\overline{\alpha}, \overline{\beta}, \overline{\gamma}), \quad \kappa = \mu\varepsilon^2\overline{\kappa}, \tag{2.12}$$

where κ is assumed to be of the order of ε^2. In order to get some feeling as to the meaning of this assumed scale of κ, it is somewhat instructive to investigate whether κ has a special meaning within the theory of micropolar elasticity. First suppose that all the microrotations, φ_k, the body couples, l_k, and the surface couples, $m_k^\mp$, are identically zero for the time being. In such a case, we notice that the equations of classical elasticity cannot be obtained from the above equations of micropolar elasticity without by setting κ equal to 0, which give a special importance to κ within four extra elastic moduli; namely, κ, α, β and γ. Then it is natural to expect that κ tends to 0 with the vanishing thickness. Furthermore, recalling that the experimental values of κ are not available and the ratio κ/μ is calculated aproximately as 0.008 by Gauthier and Jashman [14] for a composite of aliminum shot in an epoxy host, we concluded that the above scale is physically reasonable.

We close this part of the analysis with a few remarks about the choice of the scale of j. Since we choose h as the typical length scale for the micropolar effects, it is natural to introduce the non-dimensional form of j as follows $j = L^2\varepsilon^2\bar{j}$. However, with this choice of the scaling of j, the inertia terms $\ddot{\varphi}_k$ cannot be included in the lowest order approximation. On the other hand, for comparison purposes, we want to keep these terms in the lowest order approximation. Therefore, we presently assume $j = L^2\bar{j}$ instead of $j = L^2\varepsilon^2\bar{j}$. After we get the lowest order equations, in the limit $j \to 0$ the equations corresponding to the case where microinertia is negligible are recovered. Another way of keeping the inertia terms in the equations is to introduce appropriate time scales [15,16].

2.3. Transformed problem

For future reference we record here the basic equations of the three-dimensional problem in the dimensionless coordinates by using the definitions given above.

Balance of linear momentum

$$t_{\alpha\beta,\alpha} + t_{3\beta,3} + f_\beta = \varepsilon^2\ddot{u}_\beta, \qquad t_{\alpha3,\alpha} + t_{33,3} + f_3 = \ddot{u}_3. \tag{2.13}$$

Balance of angular momentum

$$m_{\alpha\beta,\alpha} + m_{3\beta,3} + \epsilon_{\alpha\beta}(t_{3\alpha} - t_{\alpha3}) + l_\beta = \varepsilon^2 j\ddot{\varphi}_\beta, \qquad m_{\alpha3,\alpha} + m_{33,3} + \varepsilon^{-2}\epsilon_{\alpha\beta}t_{\alpha\beta} + l_3 = \varepsilon^2 j\ddot{\varphi}_3. \tag{2.14}$$

Boundary conditions

$$\left.\begin{aligned} t_{3\beta} &= \mp t_\beta^\mp, \quad t_{33} = \mp t_3^\mp \\ m_{3\beta} &= \mp m_\beta^\mp, \quad m_{33} = \mp m_3^\mp \end{aligned}\right\} \text{ at } x_3 = \mp 1. \tag{2.15}$$

Stress components

$$\begin{aligned} t_{\alpha\beta} &= \lambda(u_{\gamma,\gamma} + \varepsilon^{-2}u_{3,3})\delta_{\alpha\beta} + u_{\alpha,\beta} + u_{\beta,\alpha} + \varepsilon^2\kappa(u_{\beta,\alpha} - \epsilon_{\alpha\beta}\varphi_3), \\ t_{\alpha3} &= \varepsilon^{-2}(u_{\alpha,3} + u_{3,\alpha}) + \kappa(u_{3,\alpha} + \epsilon_{\alpha\beta}\varphi_\beta), \\ t_{3\alpha} &= \varepsilon^{-2}(u_{3,\alpha} + u_{\alpha,3}) + \kappa(u_{\alpha,3} - \epsilon_{\alpha\beta}\varphi_\beta), \\ t_{33} &= \varepsilon^{-4}(\lambda + 2)u_{3,3} + \varepsilon^{-2}(\lambda u_{\gamma,\gamma} + \kappa u_{3,3}). \end{aligned} \tag{2.16}$$

Couple stress components

$$\begin{aligned} m_{\alpha\beta} &= \alpha(\varphi_{\gamma,\gamma} + \varphi_{3,3})\delta_{\alpha\beta} + \beta\varphi_{\alpha,\beta} + \gamma\varphi_{\beta,\alpha}, \qquad m_{\alpha3} = \varepsilon^{-2}\beta\varphi_{\alpha,3} + \gamma\varphi_{3,\alpha}, \\ m_{3\alpha} &= \varepsilon^{-2}\gamma\varphi_{\alpha,3} + \beta\varphi_{3,\alpha}, \qquad m_{33} = \varepsilon^{-2}\left[\alpha\varphi_{\gamma,\gamma} + (\alpha + \beta + \gamma)\varphi_{3,3}\right]. \end{aligned} \tag{2.17}$$

Note that, for convenience, we omit the bars on the non-dimensional quantities. An asymptotic expansion based on the above equations will be constructed in the following section.

3. Asymptotic expansion

3.1. Asymptotic expansion

Now recalling the dependence of the basic equations written in the dimensionless coordinates on ε, it is assumed that the displacement vector $\boldsymbol{u}$, the microrotation vector $\boldsymbol{\varphi}$, the stress tensor $\boldsymbol{t}$ and the couple stress tensor $\boldsymbol{m}$ may be expanded in powers of ε in the form

$$(\boldsymbol{u}, \boldsymbol{\varphi}, \boldsymbol{t}, \boldsymbol{m}) = \sum_{n=0}^{\infty} \left(\boldsymbol{u}^{(n)}, \boldsymbol{\varphi}^{(n)}, \boldsymbol{t}^{(n)}, \boldsymbol{m}^{(n)}\right)\varepsilon^{2n}. \tag{3.1}$$

The scaling made in Eqs. (2.6)–(2.12) and the expansion (3.1) are the main assumptions of this paper and an asymptotic expansion will be developed without any additional assumptions. It is expected that the leading terms of this expansion will identify the main features of the true solution of the three-dimensional problem. In this section, the equations corresponding to the zeroth-order approximation will be derived and a correspondence between the equations of two- and three-dimensional problems will be established.

If we substitute the asymptotic expansion (3.1) in the dimensionless form of the basic equations and equate to 0 the coefficients of powers of ε, we get a hierarchy of the basic equations to be satisfied for each order of ε. In what follows, we study the leading order equations of the asymptotic expansion in detail. To this end we do not record information about higher order terms which will not be needed later in the analysis.

3.2. Zeroth-order balance equations and boundary conditions

From (2.14) we immediately have

$$\epsilon_{\alpha\beta} t^{(0)}_{\alpha\beta} = t^{(0)}_{12} - t^{(0)}_{21} = 0. \tag{3.2}$$

That is, the zeroth-order in-plane stresses are symmetrical. Later, it will be shown that this condition is satisfied identically. Introducing (3.1) into (2.13) and (2.14), the balance equations corresponding to the zeroth-order approximation are obtained in the form:

$$t^{(0)}_{\alpha\beta,\alpha} + t^{(0)}_{3\beta,3} + f_\beta = 0, \tag{3.3}$$

$$t^{(0)}_{\alpha 3,\alpha} + t^{(0)}_{33,3} + f_3 = \ddot{u}^{(0)}_3, \tag{3.4}$$

$$m^{(0)}_{\alpha\beta,\alpha} + m^{(0)}_{3\beta,3} + \epsilon_{\alpha\beta}\left(t^{(0)}_{3\alpha} - t^{(0)}_{\alpha 3}\right) + l_\beta = j\ddot{\varphi}^{(0)}_\beta, \tag{3.5}$$

$$m^{(0)}_{\alpha 3,\alpha} + m^{(0)}_{33,3} + \epsilon_{\alpha\beta} t^{(1)}_{\alpha\beta} + l_3 = j\ddot{\varphi}^{(0)}_3. \tag{3.6}$$

With a similar process the boundary conditions (2.15) take the following form:

$$t^{(0)}_{3\beta} = \mp t^{\mp}_\beta, \quad t^{(0)}_{33} = \mp t^{\mp}_3 \quad \text{at } x_3 = \mp 1, \tag{3.7}$$

$$m_{3\beta}^{(0)} = \mp m_{\beta}^{\mp}, \quad m_{33}^{(0)} = \mp m_{3}^{\mp} \quad \text{at } x_3 = \mp 1 \tag{3.8}$$

for the zeroth-order approximation.

3.3. Zeroth-order constitutive equations

With a similar argument we obtain the following constitutive relations:

$$t_{33}^{(-2)} = (2+\lambda)u_{3,3}^{(0)}, \tag{3.9}$$

$$\begin{aligned} t_{\alpha\beta}^{(-1)} &= \lambda u_{3,3}^{(0)}\delta_{\alpha\beta}, \qquad t_{\alpha 3}^{(-1)} = u_{\alpha,3}^{(0)} + u_{3,\alpha}^{(0)}, \\ t_{3\alpha}^{(-1)} &= u_{3,\alpha}^{(0)} + u_{\alpha,3}^{(0)}, \quad t_{33}^{(-1)} = (2+\lambda)u_{3,3}^{(1)} + \lambda u_{\gamma,\gamma}^{(0)} + \kappa u_{3,3}^{(0)}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} t_{\alpha\beta}^{(0)} &= \lambda\left(u_{\gamma,\gamma}^{(0)} + u_{3,3}^{(1)}\right)\delta_{\alpha\beta} + u_{\alpha,\beta}^{(0)} + u_{\beta,\alpha}^{(0)}, \quad t_{\alpha 3}^{(0)} = u_{\alpha,3}^{(1)} + u_{3,\alpha}^{(1)} + \kappa\left(u_{3,\alpha}^{(0)} + \epsilon_{\alpha\beta}\varphi_{\beta}^{(0)}\right), \\ t_{3\alpha}^{(0)} &= u_{3,\alpha}^{(1)} + u_{\alpha,3}^{(1)} + \kappa\left(u_{\alpha,3}^{(0)} - \epsilon_{\alpha\beta}\varphi_{\beta}^{(0)}\right), \quad t_{33}^{(0)} = (2+\lambda)u_{3,3}^{(2)} + \lambda u_{\gamma,\gamma}^{(1)} + \kappa u_{3,3}^{(1)}, \end{aligned} \tag{3.11}$$

$$t_{\alpha\beta}^{(1)} = \lambda\left(u_{\gamma,\gamma}^{(1)} + u_{3,3}^{(2)}\right)\delta_{\alpha\beta} + u_{\alpha,\beta}^{(1)} + u_{\beta,\alpha}^{(1)} + \kappa\left(u_{\beta,\alpha}^{(0)} - \epsilon_{\alpha\beta}\varphi_{3}^{(0)}\right) \tag{3.12}$$

for the stresses and

$$m_{\alpha 3}^{(-1)} = \beta\varphi_{\alpha,3}^{(0)}, \quad m_{3\alpha}^{(-1)} = \gamma\varphi_{\alpha,3}^{(0)}, \quad m_{33}^{(-1)} = \alpha\varphi_{\gamma,\gamma}^{(0)} + (\alpha+\beta+\gamma)\varphi_{3,3}^{(0)}, \tag{3.13}$$

$$\begin{aligned} m_{\alpha\beta}^{(0)} &= \alpha\left(\varphi_{\gamma,\gamma}^{(0)} + \varphi_{3,3}^{(0)}\right)\delta_{\alpha\beta} + \beta\varphi_{\alpha,\beta}^{(0)} + \gamma\varphi_{\beta,\alpha}^{(0)}, \\ m_{\alpha 3}^{(0)} &= \beta\varphi_{\alpha,3}^{(1)} + \gamma\varphi_{3,\alpha}^{(0)}, \quad m_{3\alpha}^{(0)} = \gamma\varphi_{\alpha,3}^{(1)} + \beta\varphi_{3,\alpha}^{(0)}, \\ m_{33}^{(0)} &= \alpha\varphi_{\gamma,\gamma}^{(1)} + (\alpha+\beta+\gamma)\varphi_{3,3}^{(1)} \end{aligned} \tag{3.14}$$

for the couple stresses.

For future reference we record here the following relations between the transverse components of the zeroth-order stress and couple stress tensors:

$$t_{\alpha 3}^{(0)} - t_{3\alpha}^{(0)} = \kappa\left(u_{3,\alpha}^{(0)} - u_{\alpha,3}^{(0)} + 2\epsilon_{\alpha\beta}\varphi_{\beta}^{(0)}\right), \tag{3.15}$$

$$\gamma m_{\alpha 3}^{(0)} - \beta m_{3\alpha}^{(0)} = (\gamma^2 - \beta^2)\varphi_{3,\alpha}^{(0)}. \tag{3.16}$$

3.4. Zeroth-order displacement and microrotation fields

Since the expansion (3.1) suggests that the stress and couple stress components having negative index are 0, some constraints on the displacement and microrotation components are obtained. Now these constraints will be discussed in detail. The restrictions $t_{\alpha\beta}^{(-1)} = t_{33}^{(-2)} = 0$ give $u_{3,3}^{(0)} = 0$ which implies that the zeroth-order deflection is independent of x_3, i.e.,

$$u_3^{(0)} = w(x_1, x_2, t). \tag{3.17}$$

Similarly, the restrictions $t_{\alpha 3}^{(-1)} = t_{3\alpha}^{(-1)} = 0$ give $u_{\alpha,3}^{(0)} + u_{3,\alpha}^{(0)} = 0$ and consequently $u_{\alpha,3}^{(0)} = -w_{,\alpha}$ from which we deduce the in-plane components of the zeroth-order displacement field as

$$u_\alpha^{(0)} = v_\alpha - x_3 w_{,\alpha}, \tag{3.18}$$

where $v_\alpha = v_\alpha(x_1, x_2, t)$. This proves that the lowest order displacement field is a Kirchhoff–Love field. The restriction $t_{33}^{(-1)} = 0$ requires that $(2+\lambda)u_{3,3}^{(1)} + \lambda u_{\gamma,\gamma}^{(0)} = 0$, which yields

$$u_{3,3}^{(1)} = -\frac{\lambda}{\lambda+2}(v_{\gamma,\gamma} - x_3 w_{,\gamma\gamma}), \tag{3.19}$$

where (3.18) is used. The restrictions $m_{\alpha 3}^{(-1)} = m_{3\alpha}^{(-1)} = 0$ give $\varphi_{\alpha,3}^{(0)} = 0$ which implies that the in-plane components of the zeroth-order microrotation vector are independent of x_3, i.e.,

$$\varphi_\alpha^{(0)} = \psi_\alpha(x_1, x_2, t). \tag{3.20}$$

The restriction $m_{33}^{(-1)} = 0$ requires that $\alpha\varphi_{\gamma,\gamma}^{(0)} + (\alpha+\beta+\gamma)\varphi_{3,3}^{(0)} = 0$ and consequently $\varphi_{3,3}^{(0)} = -(\alpha/(\alpha+\beta+\gamma))\varphi_{\gamma,\gamma}^{(0)}$ from which we deduce the transverse component of the zeroth-order microrotation vector as

$$\varphi_3^{(0)} = \phi - \frac{\alpha}{\alpha+\beta+\gamma} x_3 \psi_{\gamma,\gamma}, \tag{3.21}$$

where $\phi = \phi(x_1, x_2, t)$.

Using the above results in (3.11) and (3.14) the stress components $t_{\alpha\beta}^{(0)}$ and the couple stress components $m_{\alpha\beta}^{(0)}$ can be expressed in the form

$$t_{\alpha\beta}^{(0)} = v_{\alpha,\beta} + v_{\beta,\alpha} + \frac{2\lambda}{\lambda+2} v_{\gamma,\gamma}\delta_{\alpha\beta} - 2x_3\left(w_{,\alpha\beta} + \frac{\lambda}{\lambda+2} w_{,\gamma\gamma}\delta_{\alpha\beta}\right), \tag{3.22}$$

$$m_{\alpha\beta}^{(0)} = \tilde{\alpha}\psi_{\gamma,\gamma}\delta_{\alpha\beta} + \beta\psi_{\alpha,\beta} + \gamma\psi_{\beta,\alpha}, \tag{3.23}$$

where $\tilde{\alpha}$ is defined as

$$\tilde{\alpha} = \alpha\left(1 - \frac{\alpha}{\alpha+\beta+\gamma}\right). \tag{3.24}$$

It is seen that the stress components $t^{(0)}_{\alpha\beta}$ vary linearly along the thickness of the plate whereas the couple stress components $m^{(0)}_{\alpha\beta}$ is uniform over the thickness. Similarly, employing the above results in (3.15), (3.16) and (3.12) we obtain the following relations:

$$t^{(0)}_{\alpha 3} - t^{(0)}_{3\alpha} = 2\kappa(w_{,\alpha} + \epsilon_{\alpha\beta}\psi_\beta), \tag{3.25}$$

$$\gamma m^{(0)}_{\alpha 3} - \beta m^{(0)}_{3\alpha} = (\gamma^2 - \beta^2)\left(\phi_{,\alpha} - \frac{\alpha}{\alpha+\beta+\gamma}x_3\psi_{\gamma,\gamma\alpha}\right), \tag{3.26}$$

$$\epsilon_{\alpha\beta}t^{(1)}_{\alpha\beta} = t^{(1)}_{12} - t^{(1)}_{21} = -\kappa\left[\epsilon_{\alpha\beta}v_{\alpha,\beta} + 2\left(\phi - \frac{\alpha}{\alpha+\beta+\gamma}x_3\psi_{\gamma,\gamma}\right)\right]. \tag{3.27}$$

We close this part of the analysis with a few remarks about our results. First we notice that the in-plane stress components $t^{(0)}_{\alpha\beta}$ and the in-plane couple stress components $m^{(0)}_{\alpha\beta}$ can be expressed in terms of w, v_α and ψ_α only, which are functions independent of x_3. Secondly, since the stress components $t^{(0)}_{\alpha 3}$, $t^{(0)}_{3\alpha}$ and $t^{(0)}_{33}$ and the couple stress components $m^{(0)}_{\alpha 3}$, $m^{(0)}_{3\alpha}$ and $m^{(0)}_{33}$ depend on higher order terms in the asymptotic expansion, they remain unknown at the present stage of the analysis. But, again, the differences $t^{(0)}_{\alpha 3} - t^{(0)}_{3\alpha}$ and $\gamma m^{(0)}_{\alpha 3} - \beta m^{(0)}_{3\alpha}$ are well defined and can be computed in terms of w, ϕ and ψ_α only. Thirdly, we observe that the evolution equations governing six unknown functions w, v_α, ϕ and ψ_α are still missing at this stage of the calculations. To complete our analysis we now derive the evolution equations and calculate the remaining stress and couple stress components.

3.5. Determination of the stress components $t^{(0)}_{3\beta}$

To compute the stress components $t^{(0)}_{3\beta}$ we first transform (3.3) into

$$t^{(0)}_{3\beta,3} = -\left(\frac{3\lambda+2}{\lambda+2}v_{\alpha,\alpha\beta} + v_{\beta,\alpha\alpha}\right) + 4\left(\frac{\lambda+1}{\lambda+2}\right)x_3 w_{,\alpha\alpha\beta} - f_\beta, \tag{3.28}$$

where (3.22) is used. Integrating this equation with respect to x_3 we obtain

$$t^{(0)}_{3\beta} = -x_3\left(\frac{3\lambda+2}{\lambda+2}v_{\alpha,\alpha\beta} + v_{\beta,\alpha\alpha}\right) + 2\left(\frac{\lambda+1}{\lambda+2}\right)x_3^2 w_{,\alpha\alpha\beta} - \int_{-1}^{x_3} f_\beta\,\mathrm{d}x_3 + F_\beta(x_1,x_2,t), \tag{3.29}$$

where $F_\beta(x_1,x_2,t)$ are arbitrary functions to be determined by the boundary conditions. Substitution of (3.29) into the boundary conditions (3.7) yields the following equations:

$$t^+_\beta(x_1,x_2,t) = -\left(\frac{3\lambda+2}{\lambda+2}v_{\alpha,\alpha\beta} + v_{\beta,\alpha\alpha}\right) + 2\left(\frac{\lambda+1}{\lambda+2}\right)w_{,\alpha\alpha\beta} - \int_{-1}^{1} f_\beta\,\mathrm{d}x_3 + F_\beta(x_1,x_2,t),$$
$$-t^-_\beta(x_1,x_2,t) = \frac{3\lambda+2}{\lambda+2}v_{\alpha,\alpha\beta} + v_{\beta,\alpha\alpha} + 2\left(\frac{\lambda+1}{\lambda+2}\right)w_{,\alpha\alpha\beta} + F_\beta(x_1,x_2,t).$$

While addition of these two equations gives the following second-order coupled partial differential equations:

$$\frac{3\lambda+2}{\lambda+2}v_{\alpha,\alpha\beta}+v_{\beta,\alpha\alpha}+\frac{1}{2}r_{\beta}=0, \tag{3.30}$$

their subtraction yields

$$F_{\beta}(x_1,x_2,t)=-2\left(\frac{\lambda+1}{\lambda+2}\right)w_{,\alpha\alpha\beta}+\frac{1}{2}\tilde{r}_{\beta}, \tag{3.31}$$

where

$$r_{\beta}(x_1,x_2,t)=t_{\beta}^{+}+t_{\beta}^{-}+\int_{-1}^{1}f_{\beta}\,\mathrm{d}x_3,\qquad \tilde{r}_{\beta}(x_1,x_2,t)=t_{\beta}^{+}-t_{\beta}^{-}+\int_{-1}^{1}f_{\beta}\,\mathrm{d}x_3. \tag{3.32}$$

Thus, using (3.30) and (3.31) in (3.29) we obtain the stress components $t_{3\beta}^{(0)}$ as follows:

$$t_{3\beta}^{(0)}=-2\left(\frac{\lambda+1}{\lambda+2}\right)(1-x_3^2)w_{,\alpha\alpha\beta}+\frac{1}{2}x_3r_{\beta}+\frac{1}{2}\tilde{r}_{\beta}-\int_{-1}^{x_3}f_{\beta}\,\mathrm{d}x_3. \tag{3.33}$$

On the other hand, a substitution of (3.33) into (3.25) gives the stress components $t_{\beta 3}^{(0)}$ in the form

$$t_{\beta 3}^{(0)}=-2\left(\frac{\lambda+1}{\lambda+2}\right)(1-x_3^2)w_{,\alpha\alpha\beta}+2\kappa\left(w_{,\beta}+\epsilon_{\beta\gamma}\psi_{\gamma}\right)+\frac{1}{2}x_3r_{\beta}+\frac{1}{2}\tilde{r}_{\beta}-\int_{-1}^{x_3}f_{\beta}\,\mathrm{d}x_3. \tag{3.34}$$

We notice that Eqs. (3.30) governing the functions v_1 and v_2 are the same as the basic equations for the stretching problem in the classical plate theory. Also one should notice that the stress components $t_{\alpha 3}^{(0)}$ and $t_{3\alpha}^{(0)}$ may be computed immediately once the functions w and ψ_{α} are known.

3.6. Determination of the stress component $t_{33}^{(0)}$

Similarly, to determine the stress component $t_{33}^{(0)}$ we rewrite (3.4) in the form

$$\begin{aligned} t_{33,3}^{(0)}=\;&2\left(\frac{\lambda+1}{\lambda+2}\right)(1-x_3^2)w_{,\alpha\alpha\gamma\gamma}-2\kappa\left(w_{,\alpha\alpha}+\epsilon_{\alpha\beta}\psi_{\beta,\alpha}\right)-\frac{1}{2}\tilde{r}_{\alpha,\alpha}+\ddot{w}-\frac{1}{2}x_3r_{\alpha,\alpha}\\ &+\int_{-1}^{x_3}f_{\alpha,\alpha}\,\mathrm{d}x_3-f_3, \end{aligned} \tag{3.35}$$

where (3.34) is used. Integrating this equation with respect to x_3 we get

$$\begin{aligned} t_{33}^{(0)}=\;&2\left(\frac{\lambda+1}{\lambda+2}\right)\left(x_3-\frac{1}{3}x_3^3\right)w_{,\alpha\alpha\gamma\gamma}+x_3\left[-2\kappa\left(w_{,\alpha\alpha}+\epsilon_{\alpha\beta}\psi_{\beta,\alpha}\right)-\frac{1}{2}\tilde{r}_{\alpha,\alpha}+\ddot{w}\right]-\frac{1}{4}x_3^2r_{\alpha,\alpha}\\ &+\int_{-1}^{x_3}\int_{-1}^{\eta}f_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta-\int_{-1}^{x_3}f_3\,\mathrm{d}x_3+F_3(x_1,x_2,t), \end{aligned} \tag{3.36}$$

where $F_3(x_1,x_2,t)$ is an arbitrary function. The use of this result in the boundary conditions (3.7) gives the following two equations:

$$t_3^+ = \frac{4}{3}\left(\frac{\lambda+1}{\lambda+2}\right)w_{,\alpha\alpha\gamma\gamma} + \left[-2\kappa\left(w_{,\alpha\alpha} + \epsilon_{\alpha\beta}\psi_{\beta,\alpha}\right) - \frac{1}{2}\tilde{r}_{\alpha,\alpha} + \ddot{w}\right] - \frac{1}{4}r_{\alpha,\alpha} + \int_{-1}^{1}\int_{-1}^{\eta} f_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta$$
$$- \int_{-1}^{1} f_3\,\mathrm{d}x_3 + F_3(x_1, x_2, t),$$
$$-t_3^- = -\frac{4}{3}\left(\frac{\lambda+1}{\lambda+2}\right)w_{,\alpha\alpha\gamma\gamma} - \left[-2\kappa\left(w_{,\alpha\alpha} + \epsilon_{\alpha\beta}\psi_{\beta,\alpha}\right) - \frac{1}{2}\tilde{r}_{\alpha,\alpha} + \ddot{w}\right] - \frac{1}{4}r_{\alpha,\alpha} + F_3(x_1, x_2, t).$$

While addition of these two equations gives the following fourth-order partial differential equation:

$$\frac{4}{3}\left(\frac{\lambda+1}{\lambda+2}\right)w_{,\alpha\alpha\gamma\gamma} - 2\kappa\left(w_{,\alpha\alpha} + \epsilon_{\alpha\beta}\psi_{\beta,\alpha}\right) + \ddot{w} - \frac{1}{2}\tilde{r}_{\alpha,\alpha} - \frac{1}{2}p + \frac{1}{2}\int_{-1}^{1}\int_{-1}^{\eta} f_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta = 0, \tag{3.37}$$

their subtraction yields

$$F_3(x_1, x_2, t) = \frac{1}{2}\tilde{p} + \frac{1}{4}r_{\alpha,\alpha} - \frac{1}{2}\int_{-1}^{1}\int_{-1}^{\eta} f_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta, \tag{3.38}$$

where

$$p(x_1, x_2, t) = t_3^+ + t_3^- + \int_{-1}^{1} f_3\,\mathrm{d}x_3, \qquad \tilde{p}(x_1, x_2, t) = t_3^+ - t_3^- + \int_{-1}^{1} f_3\,\mathrm{d}x_3. \tag{3.39}$$

Now, employing (3.38) in (3.36) the stress component $t_{33}^{(0)}$ is found as follows:

$$t_{33}^{(0)} = \frac{2}{3}\left(\frac{\lambda+1}{\lambda+2}\right)x_3(1 - x_3^2)w_{,\alpha\alpha\gamma\gamma} + \frac{1}{2}x_3 p + \frac{1}{2}\tilde{p} + \frac{1}{4}(1 - x_3^2)r_{\alpha,\alpha}$$
$$- \frac{1}{2}(1 + x_3)\int_{-1}^{1}\int_{-1}^{\eta} f_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta + \int_{-1}^{x_3}\int_{-1}^{\eta} f_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta - \int_{-1}^{x_3} f_3\,\mathrm{d}x_3. \tag{3.40}$$

We notice that, when polar effects are neglected, Eq. (3.37) reduces to the standard biharmonic equation for the bending problem in the classical plate theory. We also notice that the stress component $t_{33}^{(0)}$ may be computed once the function w is determined.

3.7. Determination of the couple stress components $m_{3\beta}^{(0)}$

If we rewrite (3.5) in the form

$$m_{3\beta,3}^{(0)} = -\left[(\tilde{\alpha} + \beta)\psi_{\alpha,\alpha\beta} + \gamma\psi_{\beta,\alpha\alpha}\right] + 2\kappa\epsilon_{\alpha\beta}\left(w_{,\alpha} + \epsilon_{\alpha\gamma}\psi_{\gamma}\right) - l_\beta + j\ddot{\psi}_\beta, \tag{3.41}$$

where (3.23) and (3.25) are used, and integrate this equation with respect to x_3, we obtain

$$m^{(0)}_{3\beta} = -x_3\Big[(\tilde{\alpha}+\beta)\psi_{\alpha,\alpha\beta} + \gamma\psi_{\beta,\alpha\alpha} - 2\kappa\epsilon_{\alpha\beta}\big(w_{,\alpha} + \epsilon_{\alpha\gamma}\psi_\gamma\big) - j\ddot{\psi}_\beta\Big] - \int_{-1}^{x_3} l_\beta\,\mathrm{d}x_3 + G_\beta(x_1,x_2,t), \tag{3.42}$$

where $G_\beta(x_1,x_2,t)$ are arbitrary functions. A substitution of this result in the boundary conditions (3.8) yields the following equations:

$$m^+_\beta = -\Big[(\tilde{\alpha}+\beta)\psi_{\alpha,\alpha\beta} + \gamma\psi_{\beta,\alpha\alpha} - 2\kappa\epsilon_{\alpha\beta}\big(w_{,\alpha} + \epsilon_{\alpha\gamma}\psi_\gamma\big) - j\ddot{\psi}_\beta\Big] - \int_{-1}^{1} l_\beta\,\mathrm{d}x_3 + G_\beta(x_1,x_2,t),$$
$$-m^-_\beta = (\tilde{\alpha}+\beta)\psi_{\alpha,\alpha\beta} + \gamma\psi_{\beta,\alpha\alpha} - 2\kappa\epsilon_{\alpha\beta}(w_{,\alpha} + \epsilon_{\alpha\gamma}\psi_\gamma) - j\ddot{\psi}_\beta + G_\beta(x_1,x_2,t).$$

While addition of these equations gives

$$(\tilde{\alpha}+\beta)\psi_{\alpha,\alpha\beta} + \gamma\psi_{\beta,\alpha\alpha} - 2\kappa\epsilon_{\alpha\beta}(w_{,\alpha} + \epsilon_{\alpha\gamma}\psi_\gamma) - j\ddot{\psi}_\beta + \frac{1}{2}s_\beta = 0, \tag{3.43}$$

it follows from their subtraction that

$$G_\beta(x_1,x_2,t) = \frac{1}{2}\tilde{s}_\beta, \tag{3.44}$$

where

$$s_\beta(x_1,x_2,t) = m^+_\beta + m^-_\beta + \int_{-1}^{1} l_\beta\,\mathrm{d}x_3, \qquad \tilde{s}_\beta(x_1,x_2,t) = m^+_\beta - m^-_\beta + \int_{-1}^{1} l_\beta\,\mathrm{d}x_3. \tag{3.45}$$

Thus, using (3.43) and (3.44) in (3.42) we obtain the couple stress components $m^{(0)}_{3\beta}$ in the form

$$m^{(0)}_{3\beta} = \frac{1}{2}x_3 s_\beta + \frac{1}{2}\tilde{s}_\beta - \int_{-1}^{x_3} l_\beta\,\mathrm{d}x_3. \tag{3.46}$$

Furthermore, if we introduce (3.46) into (3.26) we get the couple stress components $m^{(0)}_{\beta 3}$ as follows:

$$m^{(0)}_{\beta 3} = \frac{1}{\gamma}\left[(\gamma^2-\beta^2)\left(\phi_{,\beta} - \frac{\alpha}{\alpha+\beta+\gamma}x_3\psi_{\gamma,\gamma\beta}\right) + \beta\left(\frac{1}{2}x_3 s_\beta + \frac{1}{2}\tilde{s}_\beta - \int_{-1}^{x_3} l_\beta\,\mathrm{d}x_3\right)\right]. \tag{3.47}$$

We notice that Eqs. (3.37) and (3.43) couple the functions w and ψ_α through the constant κ. We also point out that Eq. (3.43) is identically satisfied if polar effects are discarded. One should notice that the couple stress components $m^{(0)}_{3\beta}$ may be computed a priori once the surface and body couples are known, without the knowledge of the other unknowns. On the other hand, the couple stress components $m^{(0)}_{\beta 3}$ are calculated once the functions ϕ and ψ_α are determined.

3.8. Determination of the couple stress component $m_{33}^{(0)}$

Similarly, to determine the couple stress component $m_{33}^{(0)}$ we rewrite (3.6) in the form

$$
\begin{aligned}
m_{33,3}^{(0)} = & -\left[\frac{1}{\gamma}(\gamma^2-\beta^2)\phi_{,\alpha\alpha}+\frac{\beta}{2\gamma}\tilde{s}_{\alpha,\alpha}-\kappa\epsilon_{\alpha\beta}v_{\alpha,\beta}-2\kappa\phi-j\ddot{\phi}\right]+x_3\left\{\frac{\alpha}{\alpha+\beta+\gamma}\left[\frac{1}{\gamma}(\gamma^2-\beta^2)\psi_{\gamma,\gamma\alpha\alpha}\right.\right. \\
& \left.\left.-2\kappa\psi_{\gamma,\gamma}-j\ddot{\psi}_{\gamma,\gamma}\right]-\frac{\beta}{2\gamma}s_{\alpha,\alpha}\right\}+\frac{\beta}{\gamma}\int_{-1}^{x_3}l_{\alpha,\alpha}\,\mathrm{d}x_3-l_3, \qquad (3.48)
\end{aligned}
$$

where (3.27) and (3.47) are used. Integrating this equation with respect to x_3 we obtain

$$
\begin{aligned}
m_{33}^{(0)} = & -x_3\left[\frac{1}{\gamma}(\gamma^2-\beta^2)\phi_{,\alpha\alpha}+\frac{\beta}{2\gamma}\tilde{s}_{\alpha,\alpha}-\kappa\epsilon_{\alpha\beta}v_{\alpha,\beta}-2\kappa\phi-j\ddot{\phi}\right] \\
& +\frac{x_3^2}{2}\left\{\frac{\alpha}{\alpha+\beta+\gamma}\left[\frac{1}{\gamma}(\gamma^2-\beta^2)\psi_{\gamma,\gamma\alpha\alpha}-2\kappa\psi_{\gamma,\gamma}-j\ddot{\psi}_{\gamma,\gamma}\right]-\frac{\beta}{2\gamma}s_{\alpha,\alpha}\right\} \\
& +\frac{\beta}{\gamma}\int_{-1}^{x_3}\int_{-1}^{\eta}l_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta-\int_{-1}^{x_3}l_3\,\mathrm{d}x_3+G_3(x_1,x_2,t), \qquad (3.49)
\end{aligned}
$$

where $G_3(x_1,x_2,t)$ is an arbitrary function. The use of this result in the boundary conditions (3.8) gives the following two equations:

$$
\begin{aligned}
m_3^+ = & -\left[\frac{1}{\gamma}(\gamma^2-\beta^2)\phi_{,\alpha\alpha}+\frac{\beta}{2\gamma}\tilde{s}_{\alpha,\alpha}-\kappa\epsilon_{\alpha\beta}v_{\alpha,\beta}-2\kappa\phi-j\ddot{\phi}\right]+\frac{1}{2}\left\{\frac{\alpha}{\alpha+\beta+\gamma}\left[\frac{1}{\gamma}(\gamma^2-\beta^2)\psi_{\gamma,\gamma\alpha\alpha}\right.\right. \\
& \left.\left.-2\kappa\psi_{\gamma,\gamma}-j\ddot{\psi}_{\gamma,\gamma}\right]-\frac{\beta}{2\gamma}s_{\alpha,\alpha}\right\}+\frac{\beta}{\gamma}\int_{-1}^{1}\int_{-1}^{\eta}l_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta-\int_{-1}^{1}l_3\,\mathrm{d}x_3+G_3(x_1,x_2,t), \\
-m_3^- = & \frac{1}{\gamma}(\gamma^2-\beta^2)\phi_{,\alpha\alpha}+\frac{\beta}{2\gamma}\tilde{s}_{\alpha,\alpha}-\kappa\epsilon_{\alpha\beta}v_{\alpha,\beta}-2\kappa\phi-j\ddot{\phi}+\frac{1}{2}\left\{\frac{\alpha}{\alpha+\beta+\gamma}\left[\frac{1}{\gamma}(\gamma^2-\beta^2)\psi_{\gamma,\gamma\alpha\alpha}\right.\right. \\
& \left.\left.-2\kappa\psi_{\gamma,\gamma}-j\ddot{\psi}_{\gamma,\gamma}\right]-\frac{\beta}{2\gamma}s_{\alpha,\alpha}\right\}+G_3(x_1,x_2,t).
\end{aligned}
$$

While addition of these two equations gives the following second-order partial differential equation:

$$
\frac{1}{\gamma}(\gamma^2-\beta^2)\phi_{,\alpha\alpha}-\kappa(2\phi+\epsilon_{\alpha\beta}v_{\alpha,\beta})-j\ddot{\phi}+\frac{1}{2}b+\frac{\beta}{2\gamma}\tilde{s}_{\alpha,\alpha}-\frac{\beta}{2\gamma}\int_{-1}^{1}\int_{-1}^{\eta}l_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta=0, \qquad (3.50)
$$

their subtraction yields

$$G_3(x_1,x_2,t) = \frac{1}{2}\tilde{b} - \frac{1}{2}\left\{\frac{\alpha}{\alpha+\beta+\gamma}\left[\frac{1}{\gamma}(\gamma^2-\beta^2)\psi_{\gamma,\gamma\alpha\alpha} - 2\kappa\psi_{\gamma,\gamma} - j\ddot{\psi}_{\gamma,\gamma}\right] - \frac{\beta}{2\gamma}s_{\alpha,\alpha}\right\}$$

$$- \frac{\beta}{2\gamma}\int_{-1}^{1}\int_{-1}^{\eta} l_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta, \tag{3.51}$$

where

$$b(x_1,x_2,t) = m_3^+ + m_3^- + \int_{-1}^{1} l_3\,\mathrm{d}x_3, \qquad \tilde{b}(x_1,x_2,t) = m_3^+ - m_3^- + \int_{-1}^{1} l_3\,\mathrm{d}x_3. \tag{3.52}$$

Now, employing (3.50) and (3.51) in (3.49) the couple stress component $m_{33}^{(0)}$ is found as follows:

$$m_{33}^{(0)} = -\frac{1}{2}(1-x_3^2)\left\{\frac{\alpha}{\alpha+\beta+\gamma}\left[\frac{1}{\gamma}(\gamma^2-\beta^2)\psi_{\gamma,\gamma\alpha\alpha} - 2\kappa\psi_{\gamma,\gamma} - j\ddot{\psi}_{\gamma,\gamma}\right] - \frac{\beta}{2\gamma}s_{\alpha,\alpha}\right\} + \frac{1}{2}x_3 b$$

$$+ \frac{1}{2}\tilde{b} - \int_{-1}^{x_3} l_3\,\mathrm{d}x_3 - \frac{\beta}{2\gamma}\left[(1+x_3)\int_{-1}^{1}\int_{-1}^{\eta} l_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta - 2\int_{-1}^{x_3}\int_{-1}^{\eta} l_{\alpha,\alpha}\,\mathrm{d}x_3\,\mathrm{d}\eta\right]. \tag{3.53}$$

Again, we notice that Eq. (3.50) couples the functions v_α and ϕ through the constant κ.

3.9. Summary

Before closing the section, let us summarize the results obtained up to now. We first observe that all the zeroth-order stresses and the zeroth-order couple stresses can be expressed in terms of the functions w, v_α, ϕ and ψ_α which are independent of x_3. The equations governing the functions w, v_α, ϕ and ψ_α are given by (3.37), (3.30), (3.50) and (3.43). An examination of these equations shows that, as in the classical plate theory, the equations governing the flexural (bending) and the extensional (stretching) motions of the plate are independent of each other. In other words, the governing equations separate into two groups, one for the extensional motions and one for the flexural motions of the plate. In particular, Eqs. (3.37) and (3.43) constitute the first group, whereas Eqs. (3.30) and (3.50) constitute the second group. The most important feature of these equations is that there is coupling between displacement and microrotation fields in each group. Furthermore, we notice that, in the absence of polar effects, these equations coincide with those of the classical plate theory.

For simplicity, in the rest of the paper, we shall restrict our analysis to the case where the body forces and the body couples vanish, i.e.,

$$f_\alpha = f_3 = l_\alpha = l_3 = 0. \tag{3.54}$$

For future use, we record the equations governing the extensional and flexural motions of the plate here:

Extensional motions

$$\frac{3\lambda+2}{\lambda+2}v_{\alpha,\alpha\beta}+v_{\beta,\alpha\alpha}+\frac{1}{2}\left(t_\beta^+ + t_\beta^-\right)=0, \tag{3.55}$$

$$\frac{1}{\gamma}(\gamma^2-\beta^2)\phi_{,\alpha\alpha}-\kappa\left(2\phi+\epsilon_{\alpha\beta}v_{\alpha,\beta}\right)-j\ddot{\phi}+\frac{1}{2}(m_3^+ + m_3^-)+\frac{\beta}{2\gamma}\left(m_{\alpha,\alpha}^+ - m_{\alpha,\alpha}^-\right)=0. \tag{3.56}$$

Flexural motions

$$\frac{4}{3}\left(\frac{\lambda+1}{\lambda+2}\right)w_{,\alpha\alpha\gamma\gamma}-2\kappa\left(w_{,\alpha\alpha}+\epsilon_{\alpha\beta}\psi_{\beta,\alpha}\right)+\ddot{w}-\frac{1}{2}(t_3^+ + t_3^-)-\frac{1}{2}\left(t_{\alpha,\alpha}^+ - t_{\alpha,\alpha}^-\right)=0, \tag{3.57}$$

$$(\tilde{\alpha}+\beta)\psi_{\alpha,\alpha\beta}+\gamma\psi_{\beta,\alpha\alpha}-2\kappa\epsilon_{\alpha\beta}\left(w_{,\alpha}+\epsilon_{\alpha\gamma}\psi_\gamma\right)-j\ddot{\psi}_\beta+\frac{1}{2}\left(m_\beta^+ + m_\beta^-\right)=0. \tag{3.58}$$

Of specific interest are the equations governing w and ψ_α, that is those for the flexure motions of the plate. In the following section various aspects of the asymptotic plate equations are discussed. In particular, it is shown how the micropolar plate equations of Eringen are related to the present theory and the speeds of flexural waves in an infinite plate are given.

4. Some aspects of the asymptotic plate equations

4.1. The case of $j = L^2\varepsilon^2\bar{j}$

In this case, the corresponding equations of micropolar plates are obtained by neglecting the terms containing j. Thus, Eqs. (3.56) and (3.58) reduce, respectively, to

$$\frac{1}{\gamma}(\gamma^2-\beta^2)\phi_{,\alpha\alpha}-\kappa(2\phi+\epsilon_{\alpha\beta}v_{\alpha,\beta})+\frac{1}{2}(m_3^+ + m_3^-)+\frac{\beta}{2\gamma}\left(m_{\alpha,\alpha}^+ - m_{\alpha,\alpha}^-\right)=0, \tag{4.1}$$

$$(\tilde{\alpha}+\beta)\psi_{\alpha,\alpha\beta}+\gamma\psi_{\beta,\alpha\alpha}-2\kappa\epsilon_{\alpha\beta}(w_{,\alpha}+\epsilon_{\alpha\gamma}\psi_\gamma)+\frac{1}{2}\left(m_\beta^+ + m_\beta^-\right)=0. \tag{4.2}$$

Furthermore, we drop the terms containing j in (3.53). Note that Eqs. (3.55) and (3.57) are not affected.

4.2. The micropolar plate equations of Eringen

As in [9], we now decompose ψ_α into a gradient and a curl as follows:

$$\psi_\alpha = \Psi_{,\alpha}+\epsilon_{\alpha\beta}\Phi_{,\beta}, \tag{4.3}$$

where Ψ and Φ are functions of x_1, x_2 and t. If we substitute Eq. (4.3) into (3.57) and (4.2) and take the divergence and curl of (4.2) we reach the following system of equations:

$$\frac{4}{3}\left(\frac{\lambda+1}{\lambda+2}\right)w_{,\alpha\alpha\gamma\gamma} - 2\kappa w_{,\alpha\alpha} + 2\kappa\Phi_{,\alpha\alpha} + \ddot{w} - \frac{1}{2}(t_3^+ + t_3^-) - \frac{1}{2}\left(t_{\alpha,\alpha}^+ - t_{\alpha,\alpha}^-\right) = 0, \tag{4.4}$$

$$\gamma\Phi_{,\alpha\alpha\beta\beta} + 2\kappa w_{,\alpha\alpha} - 2\kappa\Phi_{,\alpha\alpha} + \frac{1}{2}\epsilon_{\alpha\beta}\left(m_{\alpha,\beta}^+ + m_{\alpha,\beta}^-\right) = 0, \tag{4.5}$$

$$(\tilde{\alpha} + \beta + \gamma)\Psi_{,\alpha\alpha\beta\beta} - 2\kappa\Psi_{,\alpha\alpha} + \frac{1}{2}\left(m_{\alpha,\alpha}^+ + m_{\alpha,\alpha}^-\right) = 0. \tag{4.6}$$

The first two equations of the above system are completely equivalent to those obtained by Eringen [9], which correspond to Eqs. (7.7) and (7.8) of Eringen [9], respectively. The last equation of the above system will be in complete agreement with Eq. (7.9) of Eringen [9] provided that the following two conditions are satisfied. The first condition requires that the effect of rotatory inertia is neglected in [9]. This amounts to multiplying Eq. (7.9) of Eringen [9] by I and then to neglecting the terms that contain I as a factor. The second condition requires that the coefficient $\tilde{\alpha}$ in Eq. (4.6) of the present paper is identified with the material constant α in (7.9) of Eringen [9]. The difference between the present theory and the theory of Eringen, namely that $\tilde{\alpha}$ is replaced by α is a direct consequence of different assumptions related to the microrotation φ_3. We note that the microrotation φ_3 is a linear function of x_3 in the present paper (see (3.21)), whereas it is assumed to be independent of x_3 in [9].

4.3. Propagation of flexural waves

We now calculate the speeds of the flexural waves propagating in an infinite micropolar elastic plate. To this end we assume that the external forces and couples vanish, i.e.,

$$t_3^+ = t_3^- = t_\alpha^+ = t_\alpha^- = m_\alpha^+ = m_\alpha^- = 0 \tag{4.7}$$

and that x_1 denotes the coordinate along the direction of propagation of waves. Furthermore, we assume harmonic wave solutions for w and ψ_β in the form

$$w = w_0 \exp[\mathrm{i}(kx_1 - \omega t)], \qquad \psi_\beta = \psi_{\beta 0} \exp[\mathrm{i}(kx_1 - \omega t)], \tag{4.8}$$

where w_0 and $\psi_{\beta 0}$ are constants and k and ω denote wavenumber and circular frequency, respectively. A substitution of these solutions into (3.57) and (3.58) leads to the following set of homogeneous equations in terms of the amplitudes ψ_{10}, ψ_{20} and w_0:

$$\left[j\omega^2 - 2\kappa - (\tilde{\alpha} + \beta + \gamma)k^2\right]\psi_{10} = 0, \tag{4.9}$$

$$-2\kappa \mathrm{i}k w_0 + (j\omega^2 - 2\kappa - \gamma k^2)\psi_{20} = 0, \tag{4.10}$$

$$\left\{\omega^2 - \left[2\kappa + \frac{4}{3}\left(\frac{\lambda+1}{\lambda+2}\right)k^2\right]k^2\right\}w_0 + 2\kappa \mathrm{i}k\psi_{20} = 0. \tag{4.11}$$

For the existence of a non-trivial solution, the determinant of the coefficient matrix obtained for w_0 and ψ_{20} must vanish. Eq. (4.9) and this condition yield the following equations:

$$jc^2 = \tilde{\alpha} + \beta + \gamma + \frac{2\kappa}{k^2}, \tag{4.12}$$

$$c^4 - Ac^2 + B = 0, \tag{4.13}$$

in which c denotes the wave speed defined by $c = \omega/k$ and the coefficients A and B are given by:

$$A = 2\kappa + \frac{4}{3}\left(\frac{\lambda+1}{\lambda+2}\right)k^2 + \frac{1}{j}\left(\gamma + \frac{2\kappa}{k^2}\right), \tag{4.14}$$

$$B = \frac{1}{j}\left[2\kappa\gamma + \frac{4}{3}\left(\frac{\lambda+1}{\lambda+2}\right)\left(2\kappa + \gamma k^2\right)\right]. \tag{4.15}$$

These equations are similar in character to those given by Eringen [9]. To compare Eqs. (4.12) and (4.13) with those obtained in [9] we should neglect the transverse shear effect and the rotatory inertia effect considered in [9]. This can be done by neglecting the terms that contain G in the denominator and the terms that contain I as a factor in Eqs. (9.4) and (9.13) of Eringen [9]. In such a case, Eq. (4.13) will be in complete agreement with Eq. (9.4) of Eringen [9]. There is a misprint in Eq. (9.13) of Eringen [9] where the factor I should be present in the first term in the right-hand side. This situation is also clear from Eq. (7.2) of Eringen [9]. Thus, Eq. (4.12) will be in complete agreement with Eq. (9.13) of Eringen [9] provided that the coefficient $\tilde{\alpha}$ in Eq. (4.12) of the present paper is identified with the material constant α in Eq. (9.13) of Eringen [9].

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