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The dynamic response of an incompressible non-linearly elastic membrane tube subjected to a dynamic extension

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Abstract

The dynamic response of an isotropic hyperelastic membrane tube, subjected to a dynamic extension at its one end, is studied. In the first part of the paper, an asymptotic expansion technique is used to derive a non-linear membrane theory for finite axially symmetric dynamic deformations of incompressible non-linearly elastic circular cylindrical tubes by starting from the three-dimensional elasticity theory. The equations governing dynamic axially symmetric deformations of the membrane tube are obtained for an arbitrary form of the strain-energy function. In the second part of the paper, finite amplitude wave propagation in an incompressible hyperelastic membrane tube is considered when one end is fixed and the other is subjected to a suddenly applied dynamic extension. A Godunov-type finite volume method is used to solve numerically the corresponding problem. Numerical results are given for the Mooney–Rivlin incompressible material. The question how the present numerical results are related to those obtained in the literature is discussed.

Keywords: Nonlinear elasticity; Incompressible; Membrane tubes; Finite amplitude wave propagation

1. Introduction

The present study has several purposes. The first one is to derive a non-linear membrane theory for finite axially symmetric dynamic deformations of incompressible non-linearly elastic circular cylindrical tubes. The second purpose of the present study is to investigate finite amplitude wave propagation in a circular cylindrical non-linearly elastic membrane tube when the tube is subjected to dynamic extension.

Substantial theoretical and analytical results in the area of non-linear elasticity have been obtained in the last four decades. With regard to dynamic deformations, only a few exact solutions have been found for non-linearly elastic solids. We refer the reader to [1] for both a discussion on exact solutions and a comprehensive list of references. The difficulties in obtaining the exact solutions of non-linear elasticity problems have stimulated considerable research effort on the derivation of approximate theories which are based on the introduction of some simplifying approximations. Non-linear membrane theories among the others are

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the most common examples of these approximate theories. They are based on two fundamental assumptions: (i) "membrane stiffness is much, much greater than bending stiffness" and (ii) "the ratio of thickness to the smallest radius of curvature is much, much less than one" [2]. Approaches to membrane theories range from the purely direct to the purely derived. A direct theory regards a membrane, from the start, as a two-dimensional continuum. However, the derived approach recognizes a membrane as a three-dimensional elastic body and derives a system of two-dimensional equations from the three-dimensional elasticity theory by asymptotic analysis. Of course, there is the fundamental problem of justifying the accuracy of membrane theory solutions as approximations of the exact solutions of non-linear elasticity. We refer the reader to [3] for both a recent review on elastic membranes and a comparison of the direct and derived theories from a constitutive point of view.

The first part of the present study is devoted to the derivation of a non-linear membrane theory for incompressible non-linearly elastic circular cylindrical membranes. An asymptotic expansion technique is used to obtain the reduced equations of motion by starting from the three-dimensional elasticity theory. Much of the research in the literature has been concerned with the case in which the inner surface of the tube is subjected to only normal tractions. Here it is assumed that the inner surface of the tube is subjected to both normal and tangential time-dependent tractions, while the outer surface is free of tractions. It will be seen in the sequel that the analysis contained in the present study parallels quite closely that described in [4]. The present discussion of membrane theory is set in a framework which differs in three aspects from that presented in [4]. First, we are concerned with finite dynamic deformations of non-linearly elastic membrane tubes here, whereas the membrane theory derived in [4] is valid only for finite static deformations. Another difference between [4] and the present study arises in consideration of the tube material. Here we consider a circular cylindrical tube composed of a general incompressible hyperelastic material. On the other hand, in [4] a circular cylindrical tube made of a neo-Hookean elastic material has been considered. Finally, we prefer to use the nominal stress tensor instead of the Cauchy stress tensor used in [4].

In the second part of the present paper, the propagation of finite amplitude waves which arise in axially symmetric deformations of an incompressible hyperelastic membrane tube is considered. It is assumed that one end of the tube is fixed and that the other end is subjected to a dynamic extension. The equations of motion along with compatibility conditions are written as a quasilinear hyperbolic system of first-order partial differential equations. The same problem has been studied by Tait and Zhong [5] who have considered the membrane theory obtained using the direct approach. They have employed a numerical technique based on the method of characteristics and presented the numerical results for the Mooney–Rivlin material. In another study [6], Tait and Zhong have also considered the same tube and, in addition to the extension, imposed a dynamical twist at the moving end. They have presented the numerical results for the neo-Hookean material. The present discussion of the problem is set in a framework which differs in several aspects from that studied by Tait and Zhong [5]. First, we employ the membrane theory derived in the first part of the present study by using an asymptotic expansion technique. Second, we use a second-order Godunov method for the numerical solution of the governing equations. Furthermore, we discuss a possible existence of shock waves numerically.

The purpose of Section 2 is to briefly summarize the basic equations of finite elasticity theory. The specializations appropriate for cylindrical coordinates are also noted in Section 2. A circular cylindrical tube composed of a general homogeneous isotropic incompressible non-linearly elastic material is considered and the equations governing finite axisymmetric dynamic deformations of the tube are derived in the same section. In Section 3 it is first assumed that the thickness of the tube is much smaller than the radius of its inner surface in the undeformed configuration and then an asymptotic expansion technique is used to derive the equations governing deformations of the thin-walled tube. For this purpose, all the field variables are scaled utilizing an appropriate thickness parameter, and the non-dimensional form of the governing equations are given. Then, the deformed radial and axial coordinates, the stress tensor components, and the surface tractions are expanded into power series of the thickness parameter. The equations corresponding to the lowest-order approximation are studied in detail. It is shown that the membrane equations given in the literature are obtained as special cases of the present derivation. In Section 4 we first formulate the problem of dynamic extension for the membrane tube and then solve the corresponding problem numerically by using a second-order Godunov method. We present the numerical results for the Mooney–Rivlin strain-energy function. Also we discuss how the results obtained in the present study are related to those obtained in [5].

2. Basic equations

2.1. Preliminaries: equations of finite elasticity

Here a brief introduction to the three-dimensional basic equations of finite elasticity theory is given and the equations of motion in terms of the nominal stresses are presented in component form for a full general deformation field in cylindrical polar coordinates. More detail can be found in [1].

Consider a body that occupies a region $\mathscr{B}_0$ of space in its reference configuration or undeformed state. Suppose that the body occupies a region $\mathscr{B}$ of space at current time t. We use fixed rectangular Cartesian coordinate systems to describe the locations of particles in the body for both the reference and current states. Let $\mathbf{X}$ and $\mathbf{x}$ represent the places occupied by a material particle in the reference and current configurations, respectively, in the same Cartesian coordinate system. Henceforth, the notations Grad and Div will refer to the gradient and divergence operations, respectively, taken with respect to $\mathbf{X}$ and a superposed dot will be used to denote $\partial/\partial t$ at fixed $\mathbf{X}$. A deformation is a smooth, one-to-one map $\boldsymbol{\chi}$ that associates $\mathbf{x}$ to $\mathbf{X}$: $\mathbf{x}=\boldsymbol{\chi}(\mathbf{X},t)$. The second-order tensor $\mathbf{F}$ defined by $\mathbf{F}\equiv \operatorname{Grad}\boldsymbol{\chi}$ is called the deformation gradient and it satisfies $J\equiv\det\mathbf{F}>0$; where the notation det refers to determinant. The symmetric tensor $\mathbf{C}\equiv\mathbf{F}^{\mathrm{T}}\mathbf{F}$ is called the right Cauchy–Green deformation tensor, where the superposed T denotes the transpose. The principal invariants I_k $(k=1,2,3)$ of $\mathbf{C}$ are given by

$$I_1=\operatorname{tr}\mathbf{C}=\Lambda_1^2+\Lambda_2^2+\Lambda_3^2,\quad I_2=\tfrac{1}{2}[(\operatorname{tr}\mathbf{C})^2-\operatorname{tr}\mathbf{C}^2]=\Lambda_1^2\Lambda_2^2+\Lambda_2^2\Lambda_3^2+\Lambda_1^2\Lambda_3^2,$$
$$I_3=\det\mathbf{C}=\Lambda_1^2\Lambda_2^2\Lambda_3^2, \tag{2.1}$$

where the notation tr refers to trace and Λ_1^2, Λ_2^2 and Λ_3^2 are the eigenvalues of $\mathbf{C}$. The scalars Λ_k $(k=1,2,3)$ are also called the principal stretches [1]. Note that $I_1=I_2=3$ and $I_3=1$ in the reference configuration.

The strain-energy density per unit undeformed volume for an incompressible, isotropic and homogeneous hyperelastic material is given by $W=W(I_1,I_2)$. Since incompressibility requires that the deformation is volume preserving (isochoric), we have $I_3=J^2=1$ or, equivalently, $\Lambda_1\Lambda_2\Lambda_3=1$. Using this equation in (2.1), we can rewrite the principal invariants I_1 and I_2 in the form

$$I_1=\Lambda_1^2+\Lambda_2^2+\Lambda_1^{-2}\Lambda_2^{-2},\quad I_2=\Lambda_1^2\Lambda_2^2+\Lambda_1^{-2}+\Lambda_2^{-2}. \tag{2.2}$$

We may also regard W as a function of the principal stretches: $W=W(\Lambda_1,\Lambda_2,\Lambda_3)$, $\Lambda_1\Lambda_2\Lambda_3=1$. Note that we use the same letter to indicate the revised functional dependence. We now deduce from (2.2) that

$$\frac{\partial W}{\partial \Lambda_1}=2\Lambda_1(1-\Lambda_1^{-4}\Lambda_2^{-2})\left(\frac{\partial W}{\partial I_1}+\Lambda_2^2\frac{\partial W}{\partial I_2}\right), \tag{2.3}$$

$$\frac{\partial W}{\partial \Lambda_2}=2\Lambda_2(1-\Lambda_1^{-2}\Lambda_2^{-4})\left(\frac{\partial W}{\partial I_1}+\Lambda_1^2\frac{\partial W}{\partial I_2}\right), \tag{2.4}$$

which will be used in Section 3. The strain-energy function relates the deformation gradient tensor $\mathbf{F}$ to the nominal stress tensor $\mathbf{S}$ through the constitutive relation

$$\mathbf{S}=2\frac{\partial W}{\partial I_1}\mathbf{F}^{\mathrm{T}}+2\frac{\partial W}{\partial I_2}(I_1\mathbf{I}-\mathbf{F}^{\mathrm{T}}\mathbf{F})\mathbf{F}^{\mathrm{T}}-p\mathbf{F}^{-1}, \tag{2.5}$$

where $\mathbf{I}$ is the identity tensor and p is the unknown hydrostatic pressure associated with the incompressibility condition [1]. The relationship between the nominal stress tensor $\mathbf{S}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = J^{-1}\mathbf{FS}. \tag{2.6}$$

We observe that the symmetry of $\boldsymbol{\sigma}$, which is implied by Cauchy's second law of motion, is equivalent to

$$\mathbf{FS} = \mathbf{S}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}. \tag{2.7}$$

We now introduce a second-order tensor $\mathbf{T}$, which is called the Biot stress tensor, through the relation

$$\mathbf{T}^2 = \mathbf{SS}^{\mathrm{T}}. \tag{2.8}$$

A detailed discussion of the Biot stress tensor is given in [1]. Let $\mathbf{n}$ and $\mathbf{t}$ denote the unit (outward) normal and unit tangent vectors, respectively, for the boundary $\partial\mathscr{B}$ of the current configuration $\mathscr{B}$. Then, the normal stress component σ_{n} and the tangential stress component σ_{t} of $\boldsymbol{\sigma}$ are defined in the form

$$\sigma_{\mathrm{n}} \equiv \mathbf{n}\cdot(\boldsymbol{\sigma}(\mathbf{x})\mathbf{n}), \qquad \sigma_{\mathrm{t}} \equiv \mathbf{t}\cdot(\boldsymbol{\sigma}(\mathbf{x})\mathbf{n}), \tag{2.9}$$

where $\cdot$ denotes the scalar product.

The equation of motion, in the absence of body forces, is given by

$$\mathrm{Div}\,\mathbf{S} = \rho_0\ddot{\boldsymbol{\chi}}, \tag{2.10}$$

where ρ_0 is the mass density of the material of the body in the natural configuration. In the subsequent sections, we wish to study the equations governing axisymmetric deformations of a circular cylindrical tube of non-linearly elastic material. For this aim, we now derive the component forms, referred to cylindrical polar coordinates, of the expressions $\mathrm{Div}\,\mathbf{S} \equiv \nabla\cdot\mathbf{S}$ and $\ddot{\boldsymbol{\chi}}$ appearing in (2.10). Let (R,Θ,Z) and (r,θ,z) represent cylindrical polar coordinates associated with the undeformed and deformed configurations, respectively. Also, $(\mathbf{E}_R,\mathbf{E}_\Theta,\mathbf{E}_Z)$ and $(\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z)$ are the usual orthonormal basis vectors associated with (R,Θ,Z) and (r,θ,z), respectively. The divergence of the nominal stress tensor, $\nabla\cdot\mathbf{S}$, referred to cylindrical polar coordinates, is given by

$$\begin{aligned}\nabla\cdot\mathbf{S} = {} & \left[\frac{\partial S_{Rr}}{\partial R} + \frac{1}{R}\left(\frac{\partial S_{\Theta r}}{\partial\Theta} + S_{Rr} - S_{\Theta\theta}\frac{\partial\theta}{\partial\Theta}\right) + \frac{\partial S_{Zr}}{\partial Z} - S_{R\theta}\frac{\partial\theta}{\partial R} - S_{Z\theta}\frac{\partial\theta}{\partial Z}\right]\mathbf{e}_r \\ & + \left[\frac{\partial S_{R\theta}}{\partial R} + \frac{1}{R}\left(\frac{\partial S_{\Theta\theta}}{\partial\Theta} + S_{R\theta} + S_{\Theta r}\frac{\partial\theta}{\partial\Theta}\right) + \frac{\partial S_{Z\theta}}{\partial Z} + S_{Rr}\frac{\partial\theta}{\partial R} + S_{Zr}\frac{\partial\theta}{\partial Z}\right]\mathbf{e}_\theta \\ & + \left[\frac{\partial S_{Rz}}{\partial R} + \frac{1}{R}\left(\frac{\partial S_{\Theta z}}{\partial\Theta} + S_{Rz}\right) + \frac{\partial S_{Zz}}{\partial Z}\right]\mathbf{e}_z,\end{aligned} \tag{2.11}$$

where S_{Rr}, $S_{R\theta}$, S_{Rz}, $S_{\Theta r}$, $S_{\Theta\theta}$, $S_{\Theta z}$, S_{Zr}, $S_{Z\theta}$, S_{Zz} are the physical components of the nominal stress tensor with respect to the orthonormal bases $(\mathbf{E}_R,\mathbf{E}_\Theta,\mathbf{E}_Z)$ and $(\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z)$. The acceleration field $\ddot{\boldsymbol{\chi}}$ on the right-hand side of (2.10), referred to cylindrical coordinates, is given in the form

$$\ddot{\boldsymbol{\chi}} = \left(\frac{\partial V_r}{\partial t} - \frac{1}{r}V_\theta^2\right)\mathbf{e}_r + \left(\frac{\partial V_\theta}{\partial t} + \frac{1}{r}V_rV_\theta\right)\mathbf{e}_\theta + \left(\frac{\partial V_z}{\partial t}\right)\mathbf{e}_z, \tag{2.12}$$

where $V_r = \partial r/\partial t$, $V_\theta = \partial \theta/\partial t$, $V_z = \partial z/\partial t$. Thus, the equations of motion for the nominal stresses, referred to cylindrical coordinates, are given in component form by equating the coefficients of $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ in (2.11) to the coefficients, multiplied by ρ_0, of $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ in (2.12), respectively.

2.2. Axisymmetric deformations

We consider a hollow circular cylindrical tube of homogeneous isotropic hyperelastic material which, in its (undeformed, stress-free) natural configuration, is defined by

$$R_\mathrm{i} \leqslant R \leqslant R_\mathrm{i} + H, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad 0 \leqslant Z \leqslant L, \tag{2.13}$$

where R_i is the undeformed radius of the inner surface, H is the (uniform) thickness of the tube and L is the initial length of the tube. For future convenience, we now derive the basic equations corresponding to the case where the deformed tube is assumed to be axisymmetric. The axisymmetric deformation of the tube is defined by

$$r = r(R, Z, t), \quad \theta = \Theta, \quad z = z(R, Z, t). \tag{2.14}$$

The deformation gradient tensor $\mathbf{F}$ associated with (2.14), referred to cylindrical polar coordinates, is given by, in matrix notation,

$$\mathbf{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & 0 & \dfrac{\partial r}{\partial Z} \\ 0 & \dfrac{r}{R} & 0 \\ \dfrac{\partial z}{\partial R} & 0 & \dfrac{\partial z}{\partial Z} \end{bmatrix}. \tag{2.15}$$

Then, the incompressibility constraint takes the following form:

$$J = \frac{r}{R}\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right) = 1. \tag{2.16}$$

The corresponding right Cauchy–Green tensor $\mathbf{C}$ is

$$\mathbf{C} = \begin{bmatrix} \left(\dfrac{\partial r}{\partial R}\right)^2 + \left(\dfrac{\partial z}{\partial R}\right)^2 & 0 & \left(\dfrac{\partial r}{\partial R}\dfrac{\partial r}{\partial Z} + \dfrac{\partial z}{\partial R}\dfrac{\partial z}{\partial Z}\right) \\ 0 & \left(\dfrac{r}{R}\right)^2 & 0 \\ \left(\dfrac{\partial r}{\partial R}\dfrac{\partial r}{\partial Z} + \dfrac{\partial z}{\partial R}\dfrac{\partial z}{\partial Z}\right) & 0 & \left(\dfrac{\partial r}{\partial Z}\right)^2 + \left(\dfrac{\partial z}{\partial Z}\right)^2 \end{bmatrix}. \tag{2.17}$$

Since the principal stretches are associated with the eigenvalues of $\mathbf{C}$, we have

$$\Lambda_2^2 = \left(\frac{r}{R}\right)^2, \quad \Lambda_1^2\Lambda_3^2 = \left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right)^2$$

$$\Lambda_1^2 + \Lambda_3^2 = \left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2. \tag{2.18}$$

Thus, (2.1) gives the principal invariants I_1 and I_2 of $\mathbf{C}$ as

$$I_1 = \left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 + \left(\frac{r}{R}\right)^2,$$

$$I_2 = \left(\frac{r}{R}\right)^2 \left[\left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2\right] + \left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right)^2. \tag{2.19}$$

Similarly, (2.5) gives the physical components of the nominal stress tensor for deformation (2.14) in the form

$$S_{Rr} = 2\left(\frac{\partial W}{\partial I_1}\frac{\partial r}{\partial R} + \frac{\partial W}{\partial I_2}K_{Rr}\right) - p\,\frac{r}{R}\frac{\partial z}{\partial Z},$$

$$S_{Rz} = 2\left(\frac{\partial W}{\partial I_1}\frac{\partial z}{\partial R} + \frac{\partial W}{\partial I_2}K_{Rz}\right) + p\,\frac{r}{R}\frac{\partial r}{\partial Z},$$

$$S_{\Theta\theta} = 2\,\frac{r}{R}\left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2}K_{\Theta\theta}\right) - p\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z} - \frac{\partial z}{\partial R}\frac{\partial r}{\partial Z}\right),$$

$$S_{Zr} = 2\left(\frac{\partial W}{\partial I_1}\frac{\partial r}{\partial Z} + \frac{\partial W}{\partial I_2}K_{Zr}\right) + p\,\frac{r}{R}\frac{\partial z}{\partial R},$$

$$S_{Zz} = 2\left(\frac{\partial W}{\partial I_1}\frac{\partial z}{\partial Z} + \frac{\partial W}{\partial I_2}K_{Zz}\right) - p\,\frac{r}{R}\frac{\partial r}{\partial R},$$

$$S_{\Theta r} = S_{R\theta} = S_{Z\theta} = S_{\Theta z} = 0, \tag{2.20}$$

where $K_{Rr}, K_{Rz}, \ldots, K_{Zz}$ are given by

$$K_{Rr} = \frac{\partial r}{\partial R}\left[\left(\frac{r}{R}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2\right] - \frac{\partial r}{\partial Z}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z} + \frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right),$$

$$K_{Rz} = \frac{\partial z}{\partial R}\left[\left(\frac{r}{R}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2\right] - \frac{\partial z}{\partial Z}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z} + \frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right),$$

$$K_{\Theta\theta} = \left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2,$$

$$K_{Zr} = \frac{\partial r}{\partial Z}\left[\left(\frac{r}{R}\right)^2 + \left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2\right] - \frac{\partial r}{\partial R}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z} + \frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right),$$

$$K_{Zz} = \frac{\partial z}{\partial Z}\left[\left(\frac{r}{R}\right)^2 + \left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2\right] - \frac{\partial z}{\partial R}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z} + \frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right). \tag{2.21}$$

Then (2.6) gives the physical components of the Cauchy stress tensor $\boldsymbol{\sigma}$ in terms of the components of nominal stress tensor $\mathbf{S}$ as follows

$$\boldsymbol{\sigma} = \frac{1}{J}\begin{bmatrix} \dfrac{\partial r}{\partial R} S_{Rr} + \dfrac{\partial r}{\partial Z} S_{Zr} & 0 & \dfrac{\partial r}{\partial R} S_{Rz} + \dfrac{\partial r}{\partial Z} S_{Zz} \\ 0 & \dfrac{r}{R} S_{\Theta\theta} & 0 \\ \dfrac{\partial z}{\partial R} S_{Rr} + \dfrac{\partial z}{\partial Z} S_{Zr} & 0 & \dfrac{\partial z}{\partial R} S_{Rz} + \dfrac{\partial z}{\partial Z} S_{Zz} \end{bmatrix}. \tag{2.22}$$

We observe that, for deformation (2.14) of interest here, the symmetry property of $\boldsymbol{\sigma}$ given by (2.7) reduces to

$$\frac{\partial z}{\partial R} S_{Rr} - \frac{\partial r}{\partial R} S_{Rz} = \frac{\partial r}{\partial Z} S_{Zz} - \frac{\partial z}{\partial Z} S_{Zr}, \tag{2.23}$$

which implies that $\sigma_{rz} = \sigma_{zr}$. Using (2.8), we also observe that the eigenvalues, T_1, T_2, T_3, of the Biot stress tensor $\mathbf{T}$ must satisfy

$$T_2^2 = S_{\Theta\theta}^2, \quad T_1^2 + T_3^2 = S_{Rr}^2 + S_{Rz}^2 + S_{Zr}^2 + S_{Zz}^2, \quad T_1^2 T_3^2 = (S_{Rr} S_{Zz} - S_{Rz} S_{Zr})^2. \tag{2.24}$$

For deformation (2.14) of interest here, the general equations of motion, (2.10), reduce to

$$\frac{\partial S_{Rr}}{\partial R} + \frac{\partial S_{Zr}}{\partial Z} + \frac{1}{R}(S_{Rr} - S_{\Theta\theta}) - \rho_0 \frac{\partial^2 r}{\partial t^2} = 0, \tag{2.25}$$

$$\frac{\partial S_{Rz}}{\partial R} + \frac{\partial S_{Zz}}{\partial Z} + \frac{1}{R} S_{Rz} - \rho_0 \frac{\partial^2 z}{\partial t^2} = 0, \tag{2.26}$$

where the equations $S_{\Theta r} = S_{R\theta} = S_{Z\theta} = S_{\Theta z} = 0$ in (2.20) have been used to obtain (2.25) and (2.26).

3. An asymptotic membrane theory of hyperelastic tubes

3.1. Formulation of the problem

In this section, applying the asymptotic expansion method to the equations presented in the previous section, the equations governing the axially symmetric dynamic deformations of a membrane tube are obtained for a general form of the strain-energy function. Without any a priori assumption on the form of the field variables, it is shown that the zeroth-order approximation simultaneously include both the membrane equations previously presented in the literature and the usual assumptions on the specific forms of the field variables.

To this end we first briefly summarize the boundary conditions at the inner and outer surfaces of the tube. Consider a hollow circular cylinder made of a homogeneous, isotropic incompressible hyperelastic material, which, in its natural configuration, is defined by (2.13). The cylinder is subjected to both tangential and normal tractions on its inner surface while the outer surface is free of tractions. The surface tractions may vary axially, but not circumferentially so that the axisymmetric deformation (2.14) is produced. The inner surface of the deformed tube can be generated by rotating a continuous curve in the rz-plane about the z-axis. It is assumed that the curve does not intersect the z-axis. At time t the inner surface of the deformed tube has the parametric equations $r=r(R_i,Z,t)$, $z=z(R_i,Z,t)$. The tangential and normal unit vectors, $\mathbf{t}$ and $\mathbf{n}$, of the deformed inner surface ($R=R_i$) are

$$\mathbf{n}=-\frac{1}{\Phi}\left(\frac{\partial z}{\partial Z}\mathbf{e}_r-\frac{\partial r}{\partial Z}\mathbf{e}_z\right),\quad \mathbf{t}=\frac{1}{\Phi}\left(\frac{\partial r}{\partial Z}\mathbf{e}_r+\frac{\partial z}{\partial Z}\mathbf{e}_z\right), \tag{3.1}$$

where

$$\Phi(R,Z)\equiv\left[\left(\frac{\partial r}{\partial Z}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2\right]^{1/2}. \tag{3.2}$$

If we denote the tangential and normal tractions on the inner surface of the deformed tube by $P_t(Z,t)$ and $P_n(Z,t)$, respectively, the boundary conditions at the inner and outer surfaces of the deformed tube are given by

$$\sigma_n=-P_n,\quad \sigma_t=-P_t\quad \text{at } R=R_i \tag{3.3}$$

and

$$\sigma_n=0,\quad \sigma_t=0\quad \text{at } R=R_i+H \tag{3.4}$$

where σ_n and σ_t are the normal and tangential stress components defined by (2.9). For the present problem, using (2.9) and noting that $\sigma_{r\theta}=\sigma_{\theta r}=\sigma_{\theta z}=\sigma_{z\theta}\equiv 0$, σ_n and σ_t can be expressed in terms of the non-zero components of the Cauchy stress tensor as follows:

$$\sigma_n=\frac{1}{\Phi^2}\left[\left(\frac{\partial r}{\partial Z}\right)^2\sigma_{zz}+\left(\frac{\partial z}{\partial Z}\right)^2\sigma_{rr}-\frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z}(\sigma_{rz}+\sigma_{zr})\right] \tag{3.5}$$

and

$$\sigma_t=\frac{1}{\Phi^2}\left[\left(\frac{\partial r}{\partial Z}\right)^2\sigma_{rz}-\left(\frac{\partial z}{\partial Z}\right)^2\sigma_{zr}+\frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z}(\sigma_{zz}-\sigma_{rr})\right], \tag{3.6}$$

respectively. If we use (2.22) in (3.5) and (3.6), we can express σ_n and σ_t in terms of the non-zero components of nominal stress tensor. Introducing the resulting expression and (2.16), (2.23), (3.2) into (3.3) and (3.4), after some straightforward calculations, the boundary conditions (3.3) and (3.4) are rewritten alternatively in the form

$$S_{Rr}=-\frac{r}{R}\left(\frac{\partial z}{\partial Z}P_n-\frac{\partial r}{\partial Z}P_t\right),\quad S_{Rz}=\frac{r}{R}\left(\frac{\partial r}{\partial Z}P_n+\frac{\partial z}{\partial Z}P_t\right)\quad \text{at } R=R_i \tag{3.7}$$

and

$$S_{Rr}=0,\quad S_{Rz}=0\quad \text{at } R=R_i+H, \tag{3.8}$$

respectively. For the time being, the boundary conditions at two ends of the cylinder will not be specified. This subject will be discussed later.

3.2. Scaling

We now define the dimensionless coordinates and scale the unknowns and loads by some particular powers of an appropriate thickness parameter. The thickness parameter ε is defined in the form $\varepsilon=H/R_i$ and the thin tube assumption results in $\varepsilon\ll 1$. Consequently, we may attempt to study an asymptotic expansion of the basic equations with respect to the small dimensionless parameter ε. We now introduce the following non-dimensionalization scheme:

$$R = R_{\rm i}(1+\varepsilon\bar{R}), \quad Z = R_{\rm i}\bar{Z}, \quad r = R_{\rm i}\bar{r}, \quad z = R_{\rm i}\bar{z}, \quad t = \left(\frac{\rho_0}{\mu}\right)^{1/2} R_{\rm i}\bar{t}, \quad L = R_{\rm i}\bar{L},$$

$$(P_{\rm t}, P_{\rm n}) = \mu(\varepsilon\bar{P}_{\rm t}, \varepsilon\bar{P}_{\rm n}), \quad (S_{Rr}, S_{Rz}, S_{Zr}, S_{Zz}, S_{\Theta\theta}) = \mu(\varepsilon\bar{S}_{Rr}, \varepsilon\bar{S}_{Rz}, \bar{S}_{Zr}, \bar{S}_{Zz}, \bar{S}_{\Theta\theta})$$

$$W = \mu\bar{W}, \quad T_k = \mu\bar{T}_k \quad (k=1,2,3), \quad p = \mu\bar{p}, \tag{3.9}$$

where μ is the shear modulus for infinitesimal deformation from the undeformed state, and the common factor of the stress dimension is chosen as μ. The nominal stress components S_{Rr} and S_{Rz} are scaled such that they are included in the lowest-order approximation of (3.7). Thus, the domain occupied by the thin tube in the undeformed configuration is transformed into a domain of comparable dimensions, which is independent of ε. Since such a transformation introduces the thickness parameter ε into the field equations, the solution of the three-dimensional problem will depend not only on $\bar{R}$, $\bar{Z}$, $\bar{t}$ but also on ε. It is reasonable then to expect the solution of the corresponding problem defined in the dimensionless coordinates to have an expansion with respect to ε.

3.3. *Transformed problem*

For future reference we record here the basic equations of the three-dimensional problem in the dimensionless coordinates by using the definitions given in (3.9). Henceforth non-dimensional variables are used but the superposed bars are omitted for convenience.

The incompressibility condition (2.16) takes the following form:

$$J = \frac{r}{1+\varepsilon R}\left[\frac{1}{\varepsilon}\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right)\right] = 1. \tag{3.10}$$

The scaled form of the right Cauchy–Green tensor $\mathbf{C}$, given by (2.17), is

$$\mathbf{C} = \begin{bmatrix} \frac{1}{\varepsilon^2}\left(\frac{\partial r}{\partial R}\right)^2 + \frac{1}{\varepsilon^2}\left(\frac{\partial z}{\partial R}\right)^2 & 0 & \frac{1}{\varepsilon}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z} + \frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right) \\ 0 & \left(\frac{r}{1+\varepsilon R}\right)^2 & 0 \\ \frac{1}{\varepsilon}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z} + \frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right) & 0 & \left(\frac{\partial r}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 \end{bmatrix}. \tag{3.11}$$

Furthermore, in the dimensionless coordinates, the principal stretches Λ_k $(k=1,2,3)$, which satisfy (2.18), and the principal invariants I_k $(k=1,2)$, given by (2.19), take the following form:

$$\Lambda_2^2 = \left(\frac{r}{1+\varepsilon R}\right)^2, \quad \Lambda_1^2\Lambda_3^2 = \frac{1}{\varepsilon^2}\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z} - \frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right)^2,$$

$$\Lambda_1^2 + \Lambda_3^2 = \frac{1}{\varepsilon^2}\left[\left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial z}{\partial R}\right)^2\right] + \left(\frac{\partial r}{\partial Z}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2 \tag{3.12}$$

and

$$I_1=\frac{1}{\varepsilon^2}\left[\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial z}{\partial R}\right)^2\right]+\left(\frac{\partial r}{\partial Z}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2+\left(\frac{r}{1+\varepsilon R}\right)^2,$$

$$I_2=\left(\frac{r}{1+\varepsilon R}\right)^2\left[\frac{1}{\varepsilon^2}\left[\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial z}{\partial R}\right)^2\right]+\left(\frac{\partial r}{\partial Z}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2\right]+\frac{1}{\varepsilon^2}\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z}-\frac{\partial r}{\partial Z}\frac{\partial z}{\partial R}\right)^2, \tag{3.13}$$

respectively. It follows from (2.20) that the non-zero components of the nominal stress tensor are given by

$$S_{Rr}=2\left(\frac{1}{\varepsilon^2}\frac{\partial W}{\partial I_1}\frac{\partial r}{\partial R}+\frac{1}{\varepsilon}\frac{\partial W}{\partial I_2}K_{Rr}\right)-\frac{p}{\varepsilon}\frac{r}{1+\varepsilon R}\frac{\partial z}{\partial Z},$$

$$S_{Rz}=2\left(\frac{1}{\varepsilon^2}\frac{\partial W}{\partial I_1}\frac{\partial z}{\partial R}+\frac{1}{\varepsilon}\frac{\partial W}{\partial I_2}K_{Rz}\right)+\frac{p}{\varepsilon}\frac{r}{1+\varepsilon R}\frac{\partial r}{\partial Z},$$

$$S_{\Theta\theta}=2\frac{r}{1+\varepsilon R}\left(\frac{\partial W}{\partial I_1}+\frac{\partial W}{\partial I_2}K_{\Theta\theta}\right)-\frac{p}{\varepsilon}\left(\frac{\partial r}{\partial R}\frac{\partial z}{\partial Z}-\frac{\partial z}{\partial R}\frac{\partial r}{\partial Z}\right),$$

$$S_{Zr}=2\left(\frac{\partial W}{\partial I_1}\frac{\partial r}{\partial Z}+\frac{\partial W}{\partial I_2}K_{Zr}\right)+\frac{p}{\varepsilon}\frac{r}{1+\varepsilon R}\frac{\partial z}{\partial R},$$

$$S_{Zz}=2\left(\frac{\partial W}{\partial I_1}\frac{\partial z}{\partial Z}+\frac{\partial W}{\partial I_2}K_{Zz}\right)-\frac{p}{\varepsilon}\frac{r}{1+\varepsilon R}\frac{\partial r}{\partial R}, \tag{3.14}$$

where the following notation is used:

$$K_{Rr}=\frac{1}{\varepsilon}\frac{\partial r}{\partial R}\left[\left(\frac{r}{1+\varepsilon R}\right)^2+\left(\frac{\partial r}{\partial Z}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2\right]-\frac{1}{\varepsilon}\frac{\partial r}{\partial Z}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}+\frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right),$$

$$K_{Rz}=\frac{1}{\varepsilon}\frac{\partial z}{\partial R}\left[\left(\frac{r}{1+\varepsilon R}\right)^2+\left(\frac{\partial r}{\partial Z}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2\right]-\frac{1}{\varepsilon}\frac{\partial z}{\partial Z}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}+\frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right),$$

$$K_{\Theta\theta}=\left(\frac{\partial r}{\partial Z}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2+\frac{1}{\varepsilon^2}\left[\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial z}{\partial R}\right)^2\right],$$

$$K_{Zr}=\frac{\partial r}{\partial Z}\left\{\left(\frac{r}{1+\varepsilon R}\right)^2+\frac{1}{\varepsilon^2}\left[\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial z}{\partial R}\right)^2\right]\right\}-\frac{1}{\varepsilon^2}\frac{\partial r}{\partial R}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}+\frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right),$$

$$K_{Zz}=\frac{\partial z}{\partial Z}\left\{\left(\frac{r}{1+\varepsilon R}\right)^2+\frac{1}{\varepsilon^2}\left[\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial z}{\partial R}\right)^2\right]\right\}-\frac{1}{\varepsilon^2}\frac{\partial z}{\partial R}\left(\frac{\partial r}{\partial R}\frac{\partial r}{\partial Z}+\frac{\partial z}{\partial R}\frac{\partial z}{\partial Z}\right). \tag{3.15}$$

Furthermore, in the dimensionless coordinates, the equations of motion given by (2.25) and (2.26) reduce to

$$\frac{\partial S_{Rr}}{\partial R}+\frac{\partial S_{Zr}}{\partial Z}+\frac{1}{1+\varepsilon R}(\varepsilon S_{Rr}-S_{\Theta\theta})=\frac{\partial^2 r}{\partial t^2}, \tag{3.16}$$

$$\frac{\partial S_{Rz}}{\partial R}+\frac{\partial S_{Zz}}{\partial Z}+\frac{\varepsilon}{1+\varepsilon R}S_{Rz}=\frac{\partial^2 z}{\partial t^2}. \tag{3.17}$$

With a similar process, the scaled forms of the boundary conditions given by (3.7) and (3.8) are obtained in the form

$$S_{Rr} = -\frac{r}{1+\varepsilon R}\left(\frac{\partial z}{\partial Z}P_{\rm n} - \frac{\partial r}{\partial Z}P_{\rm t}\right),$$

$$S_{Rz} = \frac{r}{1+\varepsilon R}\left(\frac{\partial r}{\partial Z}P_{\rm n} + \frac{\partial z}{\partial Z}P_{\rm t}\right) \quad \text{at } R=0 \tag{3.18}$$

and

$$S_{Rr} = 0, \quad S_{Rz} = 0 \quad \text{at } R = 1. \tag{3.19}$$

Finally, we want to point out that the relationship given by (2.24) becomes

$$T_2^2 = S_{\Theta\theta}^2, \quad T_1^2 + T_3^2 = S_{Zr}^2 + S_{Zz}^2 + \varepsilon^2(S_{Rr}^2 + S_{Rz}^2), \quad T_1^2 T_3^2 = \varepsilon^2(S_{Rr}S_{Zz} - S_{Rz}S_{Zr})^2. \tag{3.20}$$

3.4. Asymptotic expansion

Now, recalling the dependence of the basic equations written in the dimensionless coordinates on ε, it is assumed that the deformed radial coordinate r, the deformed axial coordinate z, the nominal stress tensor $\mathbf{S}$, and the hydrostatic pressure p may be expanded in powers of ε in the form

$$r(R,Z,t) = r^{(0)}(R,Z,t) + \varepsilon r^{(1)}(R,Z,t) + \cdots, \tag{3.21}$$

$$z(R,Z,t) = z^{(0)}(R,Z,t) + \varepsilon z^{(1)}(R,Z,t) + \cdots, \tag{3.22}$$

$$\mathbf{S}(R,Z,t) = \mathbf{S}^{(0)}(R,Z,t) + \varepsilon \mathbf{S}^{(1)}(R,Z,t) + \cdots, \tag{3.23}$$

$$p(R,Z,t) = p^{(0)}(R,Z,t) + \varepsilon p^{(1)}(R,Z,t) + \cdots, \tag{3.24}$$

where the coefficients of the parameter ε are assumed to be of order unity. The scaling made in (3.9) and expansion (3.21)–(3.24) are the main assumptions of this section and an asymptotic expansion will be developed without any additional assumptions. But at this point, note that the asymptotic expansion method is not free from a priori assumptions due to the scaling assertions. It is expected that the leading terms of this expansion will identify the main features of the true solution of the three-dimensional problem.

If we substitute the asymptotic expansion (3.21)–(3.24) in the dimensionless form of the basic equations and equate the coefficients of powers of ε to zero, we get a hierarchy of the basic equations to be satisfied for each order of ε. In what follows, we study the leading order equations of the asymptotic expansion in detail. To this end we do not record information about higher-order terms which will not be needed later in the analysis.

Suppose that the expansions of $r(R,Z,t)$ and $z(R,Z,t)$ are substituted into the scaled form of the right Cauchy–Green tensor given by (3.11) and that the limits of the resulting components of the right Cauchy–Green tensor are taken as ε tends to zero. Since we require that the components of the right Cauchy–Green tensor $\mathbf{C}$ take finite values, we obtain the following equations:

$$\left(\frac{\partial r^{(0)}}{\partial R}\right)^2 + \left(\frac{\partial z^{(0)}}{\partial R}\right)^2 = 0, \quad \frac{\partial r^{(0)}}{\partial R}\frac{\partial r^{(0)}}{\partial Z} + \frac{\partial z^{(0)}}{\partial R}\frac{\partial z^{(0)}}{\partial Z} = 0. \tag{3.25}$$

The first equation implies that the derivatives of $r^{(0)}$ and $z^{(0)}$ with respect to R must vanish, i.e.

$$\frac{\partial r^{(0)}}{\partial R} \equiv 0, \quad \frac{\partial z^{(0)}}{\partial R} \equiv 0, \tag{3.26}$$

in which the second equation is satisfied identically. Now, for the sake of its simplicity in the subsequent analysis, the following notation is introduced:

$$r^{(0)} \equiv g(Z,t), \quad z^{(0)} \equiv f(Z,t). \tag{3.27}$$

Henceforth, prime denotes differentiation with respect to the dimensionless axial coordinate Z.

If we substitute the asymptotic expansion (3.21)–(3.24) in the dimensionless form of the basic equations, we obtain the function Φ, the Jacobian J, the principal stretches Λ_k $(k=1,2,3)$, the principal invariants I_k $(k=1,2)$, and the principal Biot stresses T_k $(k=1,2,3)$, given by (3.2), (3.10), (3.12), (3.13) and(3.20), respectively, as power series in ε:

$$\begin{aligned} &\Phi(R,Z,t) = \phi(Z,t) + \mathcal{O}(\varepsilon), \quad J = j + \mathcal{O}(\varepsilon), \\ &\Lambda_k = \lambda_k + \mathcal{O}(\varepsilon), \quad I_k = i_k + \mathcal{O}(\varepsilon), \quad T_k = T_k^{(0)} + \mathcal{O}(\varepsilon), \end{aligned} \tag{3.28}$$

where the zeroth-order approximation, ϕ, of the function Φ is

$$\phi = [(f')^2 + (g')^2]^{1/2}, \tag{3.29}$$

the zeroth-order principal stretch ratios λ_k $(k=1,2,3)$ satisfy

$$\lambda_2^2 = g^2, \quad \lambda_1^2 + \lambda_3^2 = \phi^2 + \left(\frac{\partial r^{(1)}}{\partial R}\right)^2 + \left(\frac{\partial z^{(1)}}{\partial R}\right)^2, \quad \lambda_1^2\lambda_3^2 = \left(f'\frac{\partial r^{(1)}}{\partial R} - g'\frac{\partial z^{(1)}}{\partial R}\right)^2, \tag{3.30}$$

the zeroth-order principal invariants i_k $(k=1,2)$ are

$$i_1 = g^2 + \phi^2 + \left(\frac{\partial r^{(1)}}{\partial R}\right)^2 + \left(\frac{\partial z^{(1)}}{\partial R}\right)^2, \quad i_2 = \frac{1}{g^2} + g^2\left[\phi^2 + \left(\frac{\partial r^{(1)}}{\partial R}\right)^2 + \left(\frac{\partial z^{(1)}}{\partial R}\right)^2\right], \tag{3.31}$$

and the zeroth-order principal Biot stresses are

$$T_1^{(0)} = \sqrt{(S_{Zr}^{(0)})^2 + (S_{Zz}^{(0)})^2}, \quad T_2^{(0)} = S_{\Theta\theta}^{(0)}, \quad T_3^{(0)} = 0. \tag{3.32}$$

Note that the incompressibility condition (3.10) takes the following form:

$$j = g\left(f'\frac{\partial r^{(1)}}{\partial R} - g'\frac{\partial z^{(1)}}{\partial R}\right) = 1 \tag{3.33}$$

at the zeroth-order approximation.

Introducing (3.21)–(3.24) into (3.16) and (3.17), the equations of motion corresponding to the zeroth-order approximation are obtained in the form

$$\frac{\partial S_{Rr}^{(0)}}{\partial R} + \frac{\partial S_{Zr}^{(0)}}{\partial Z} - S_{\Theta\theta}^{(0)} = \frac{\partial^2 g}{\partial t^2}, \quad \frac{\partial S_{Rz}^{(0)}}{\partial R} + \frac{\partial S_{Zz}^{(0)}}{\partial Z} = \frac{\partial^2 f}{\partial t^2}. \tag{3.34}$$

With a similar process, using (3.21)–(3.24) in (3.18)–(3.19), the boundary conditions at the zeroth-order approximation are found as

$$S_{Rr}^{(0)} = g(g'P_{\mathrm{t}} - f'P_{\mathrm{n}}), \quad S_{Rz}^{(0)} = g(f'P_{\mathrm{t}} + g'P_{\mathrm{n}}) \quad \text{at } R=0 \tag{3.35}$$

and

$$S_{Rr}^{(0)} = 0, \quad S_{Rz}^{(0)} = 0 \quad \text{at } R = 1. \tag{3.36}$$

In order to find the constitutive equations corresponding to the zeroth-order approximation we first expand the derivatives of the strain-energy function with respect to the principal invariants, $\partial W/\partial I_l$ $(l = 1, 2)$, appearing in the constitutive relations (3.14). For this purpose we use Taylor's expansion of a general function $F(I_1(\varepsilon), I_2(\varepsilon))$, $F(I_1(\varepsilon), I_2(\varepsilon)) = F(i_1, i_2) + \mathcal{O}(\varepsilon)$, and obtain

$$\frac{\partial W}{\partial I_l} = W_l + \mathcal{O}(\varepsilon) \quad (l = 1, 2), \tag{3.37}$$

where, for the sake of brevity, the notation

$$W_l = \left.\frac{\partial W}{\partial I_l}\right|_{I_k = i_k} \qquad (l = 1, 2) \ (k = 1, 2) \tag{3.38}$$

is used. Substituting the above equations into (3.14), the following constitutive relations for $S_{\Theta\theta}$, S_{Zr} and S_{Zz} are obtained at the zeroth-order approximation:

$$S_{\Theta\theta}^{(0)} = 2g\left\{W_1 + W_2\left[\phi^2 + \left(\frac{\partial r^{(1)}}{\partial R}\right)^2 + \left(\frac{\partial z^{(1)}}{\partial R}\right)^2\right]\right\} - \frac{1}{g}\, p^{(0)}, \tag{3.39}$$

$$S_{Zr}^{(0)} = 2g' W_1 - 2W_2\left\{\frac{\partial r^{(1)}}{\partial R}\left(g' \frac{\partial r^{(1)}}{\partial R} + f' \frac{\partial z^{(1)}}{\partial R}\right)\right.$$
$$\left. -\ g'\left[g^2 + \left(\frac{\partial r^{(1)}}{\partial R}\right)^2 + \left(\frac{\partial z^{(1)}}{\partial R}\right)^2\right]\right\} + g p^{(0)} \frac{\partial z^{(1)}}{\partial R}, \tag{3.40}$$

$$S_{Zz}^{(0)} = 2f' W_1 - 2W_2\left\{\frac{\partial z^{(1)}}{\partial R}\left(g' \frac{\partial r^{(1)}}{\partial R} + f' \frac{\partial z^{(1)}}{\partial R}\right)\right.$$
$$\left. -\ f'\left[g^2 + \left(\frac{\partial r^{(1)}}{\partial R}\right)^2 + \left(\frac{\partial z^{(1)}}{\partial R}\right)^2\right]\right\} - g p^{(0)} \frac{\partial r^{(1)}}{\partial R}. \tag{3.41}$$

Also we observe that the expansions corresponding to S_{Rr} and S_{Rz} include the negative powers of ε. Since expansion (3.23) suggests that the nominal stress tensor components having negative index are zero, some constraints on the deformed radial and axial coordinates are obtained. The restrictions $S_{Rr}^{(-1)} = 0$ and $S_{Rz}^{(-1)} = 0$ give

$$W_1 \frac{\partial r^{(1)}}{\partial R} + W_2\left[(g^2 + \phi^2)\frac{\partial r^{(1)}}{\partial R} - g'\left(g' \frac{\partial r^{(1)}}{\partial R} + f' \frac{\partial z^{(1)}}{\partial R}\right)\right] - \frac{1}{2} f' g p^{(0)} = 0 \tag{3.42}$$

and

$$W_1 \frac{\partial z^{(1)}}{\partial R} + W_2\left[(g^2 + \phi^2)\frac{\partial z^{(1)}}{\partial R} - f'\left(g' \frac{\partial r^{(1)}}{\partial R} + f' \frac{\partial z^{(1)}}{\partial R}\right)\right] + \frac{1}{2} g' g p^{(0)} = 0, \tag{3.43}$$

respectively. Multiplying these equations by f' and g', and subtracting and adding the resulting equations, we obtain

$$\left(g' \frac{\partial r^{(1)}}{\partial R} + f' \frac{\partial z^{(1)}}{\partial R}\right)\left[W_2\left(f' \frac{\partial r^{(1)}}{\partial R} - g' \frac{\partial z^{(1)}}{\partial R}\right) - \frac{1}{2} g p^{(0)}\right] = 0 \tag{3.44}$$

and

$$\left(f'\frac{\partial r^{(1)}}{\partial R}-g'\frac{\partial z^{(1)}}{\partial R}\right)[W_1+W_2(g^2+\phi^2)]-\frac{1}{2}g\phi^2p^{(0)}=0. \tag{3.45}$$

Note that the incompressibility condition (3.33) and (3.44)–(3.45) give sufficient relations to determine the unknown functions $\partial r^{(1)}/\partial R, \partial z^{(1)}/\partial R$ and $p^{(0)}$. Introducing the incompressibility condition (3.33) into (3.45) we obtain

$$p^{(0)}=\frac{2}{g^2\phi^2}[W_1+W_2(g^2+\phi^2)]. \tag{3.46}$$

Then, substituting this result and (3.33) into (3.44), we get

$$\frac{1}{g\phi^2}\left(g'\frac{\partial r^{(1)}}{\partial R}+f'\frac{\partial z^{(1)}}{\partial R}\right)(W_1+W_2g^2)=0, \tag{3.47}$$

which requires either $W_1+W_2g^2=0$ or $g'\partial r^{(1)}/\partial R+f'\partial z^{(1)}/\partial R=0$. Since we assume that W is an arbitrary function of the principal invariants, we require that the latter equation is satisfied identically. Thus, it follows from this equation and (3.33) that

$$\frac{\partial r^{(1)}}{\partial R}=\frac{f'}{g\phi^2},\qquad \frac{\partial z^{(1)}}{\partial R}=-\frac{g'}{g\phi^2}. \tag{3.48}$$

Note that in the above treatment we have not placed any restriction on the form of the strain-energy function. Eqs. (3.48) imply that $\partial r^{(1)}/\partial R$ and $\partial z^{(1)}/\partial R$ are independent of the dimensionless undeformed radial coordinate R. This shows that the first-order approximations of the deformed radial and axial coordinates, $r^{(1)}$ and $z^{(1)}$, vary linearly along the thickness of the tube, whereas the zeroth-order deformed radial and axial coordinates, g and f, are uniform over the thickness.

Substituting (3.48) into (3.30) we get

$$\lambda_2^2=g^2,\qquad \lambda_1^2+\lambda_3^2=\phi^2+\frac{1}{g^2\phi^2},\qquad \lambda_1^2\lambda_3^2=\frac{1}{g^2}, \tag{3.49}$$

from which we deduce that

$$\lambda_1=\phi,\qquad \lambda_2=g,\qquad \lambda_3=\frac{1}{g\phi}. \tag{3.50}$$

Using (3.48) in (3.31) we obtain the zeroth-order principal invariants in the form

$$i_1=\phi^2+g^2+\frac{1}{g^2\phi^2},\qquad i_2=\frac{1}{\phi^2}+\frac{1}{g^2}+g^2\phi^2. \tag{3.51}$$

By using (3.46) and (3.48) in (3.39)–(3.41), the zeroth-order components of the nominal stress tensor are obtained in the form

$$S_{\Theta\theta}^{(0)}=2g\left(1-\frac{1}{g^4\phi^2}\right)(W_1+W_2\phi^2), \tag{3.52}$$

$$S_{Zr}^{(0)}=2g'\left(1-\frac{1}{g^2\phi^4}\right)(W_1+W_2g^2), \tag{3.53}$$

$$S_{Zz}^{(0)}=2f'\left(1-\frac{1}{g^2\phi^4}\right)(W_1+W_2g^2) \tag{3.54}$$

from which we deduce

$$f' S_{Zr}^{(0)} = g' S_{Zz}^{(0)}. \tag{3.55}$$

Note that the above zeroth-order components of the nominal stress tensor are independent of R.

Integrating (3.34) with respect to R between the limits zero and one and using the boundary conditions (3.35) and (3.36), we obtain

$$\frac{\partial S_{Zr}^{(0)}}{\partial Z} - S_{\Theta\theta}^{(0)} + g(f' P_{\rm n} - g' P_{\rm t}) = \frac{\partial^2 g}{\partial t^2}, \tag{3.56}$$

$$\frac{\partial S_{Zz}^{(0)}}{\partial Z} - g(g' P_{\rm n} + f' P_{\rm t}) = \frac{\partial^2 f}{\partial t^2}, \tag{3.57}$$

from which we easily deduce

$$f' \frac{\partial S_{Zr}^{(0)}}{\partial Z} - f' S_{\Theta\theta}^{(0)} + g\phi^2 P_{\rm n} - g' \frac{\partial S_{Zz}^{(0)}}{\partial Z} = f' \frac{\partial^2 g}{\partial t^2} - g' \frac{\partial^2 f}{\partial t^2}, \tag{3.58}$$

$$g' \frac{\partial S_{Zr}^{(0)}}{\partial Z} - g' S_{\Theta\theta}^{(0)} - g\phi^2 P_{\rm t} + f' \frac{\partial S_{Zz}^{(0)}}{\partial Z} = g' \frac{\partial^2 g}{\partial t^2} + f' \frac{\partial^2 f}{\partial t^2}. \tag{3.59}$$

If the zeroth-order constitutive relations given by (3.52)–(3.54) are employed in (3.58)–(3.59), the following system of two coupled second-order non-linear partial differential equations is obtained:

$$\begin{aligned} f' g'' - g' f'' - \Bigg[& \frac{f'\phi^2}{g}(1 - g^4\phi^2)(W_1 + W_2\phi^2) + \frac{g^3\phi^6}{2} P_{\rm n} \\ & - \frac{g^2\phi^4}{2}\left(f' \frac{\partial^2 g}{\partial t^2} - g' \frac{\partial^2 f}{\partial t^2} \right) \Bigg] \frac{1}{(1 - g^2\phi^4)(W_1 + W_2 g^2)} = 0, \end{aligned} \tag{3.60}$$

$$\begin{aligned} g' g'' + f' f'' + \Bigg[& \frac{g'\phi^2}{g}[W_1(3 - g^4\phi^2) + W_2(1 + g^4\phi^2)\phi^2] \\ & - \phi^2(1 - g^2\phi^4)(W_1' + g^2 W_2') - \frac{g^3\phi^6}{2} P_{\rm t} \\ & - \frac{g^2\phi^4}{2}\left(g' \frac{\partial^2 g}{\partial t^2} + f' \frac{\partial^2 f}{\partial t^2} \right) \Bigg] \frac{1}{(3 + g^2\phi^4)(W_1 + W_2 g^2)} = 0. \end{aligned} \tag{3.61}$$

Eqs. (3.60) and (3.61) are in agreement with those derived in the membrane theory for finite static deformations. For instance, they reduce to Eqs. (3.37) of [4] when static deformations of membrane tubes made of neo-Hookean material are considered.

The above equations may also be written in terms of the principal stretch ratios and the principal Biot stresses. For this purpose, we now introduce a new function $\beta(Z,t)$ in the form

$$f' = \phi \sin\beta, \quad g' = \phi\cos\beta, \tag{3.62}$$

where $\beta(Z,t)$ is the angle measured from the outward radius vector to the tangent of the deformed interior generating line. Eq. (3.55) may now be written as

$$S_{Zz}^{(0)} = S_{Zr}^{(0)} \tan\beta. \tag{3.63}$$

From (3.63) and (3.32) we then have

$$S_{\Theta\theta}^{(0)} = T_2^{(0)}, \quad S_{Zr}^{(0)} = T_1^{(0)}\cos\beta, \quad S_{Zz}^{(0)} = T_1^{(0)}\sin\beta. \tag{3.64}$$

Thus, introducing (3.64) into (3.56) and (3.57), the equations of motion can be written in the form

$$\frac{\partial}{\partial Z}(T_1^{(0)}\cos\beta) - T_2^{(0)} + g\phi(P_{\rm n}\sin\beta - P_{\rm t}\cos\beta) = \frac{\partial^2 g}{\partial t^2}, \tag{3.65}$$

$$\frac{\partial}{\partial Z}(T_1^{(0)}\sin\beta) - g\phi(P_{\rm n}\cos\beta - P_{\rm t}\sin\beta) = \frac{\partial^2 f}{\partial t^2}. \tag{3.66}$$

We emphasize that we are not aware of any previous derivation of these equations through an asymptotic expansion technique and that we have been unable to find these equations explicitly in the literature. These equations are in agreement with those presented in the literature when static deformations are considered. For instance, in the absence of both the inertia terms and the tangential traction component, i.e. $P_{\rm t} \equiv 0$, they are identical to Eqs. (4.11.10) of [7].

In the remaining part of this section we want to express the principal Biot stress components, $T_1^{(0)}$ and $T_2^{(0)}$, appearing in (3.65) and (3.66) in terms of the derivatives of the strain-energy function with respect to the principal stretch ratios. To this end we first point out that the Taylor expansion of a general function $F(\Lambda_1(\varepsilon), \Lambda_2(\varepsilon), \Lambda_3(\varepsilon))$ is $F(\Lambda_1(\varepsilon), \Lambda_2(\varepsilon), \Lambda_3(\varepsilon)) = F(\lambda_1, \lambda_2, \lambda_3) + \mathcal{O}(\varepsilon)$, from which it may be deduced that

$$\frac{\partial W}{\partial \Lambda_l} = \left.\frac{\partial W}{\partial \Lambda_l}\right|_{\Lambda_k=\lambda_k} + \mathcal{O}(\varepsilon) \quad (l = 1,2) \ (k = 1,2,3). \tag{3.67}$$

We also observe that, at the zeroth-order approximation, (2.3) and (2.4) take the following form:

$$\left.\frac{\partial W}{\partial \Lambda_1}\right|_{\Lambda_k=\lambda_k} = 2\lambda_1(1 - \lambda_1^{-4}\lambda_2^{-2})(W_1 + \lambda_2^2 W_2), \tag{3.68}$$

$$\left.\frac{\partial W}{\partial \Lambda_2}\right|_{\Lambda_k=\lambda_k} = 2\lambda_2(1 - \lambda_1^{-2}\lambda_2^{-4})(W_1 + \lambda_1^2 W_2), \tag{3.69}$$

respectively. Then, using these equations in (3.52)–(3.54) we obtain

$$S_{\Theta\theta}^{(0)} = \left.\frac{\partial W}{\partial \Lambda_2}\right|_{\Lambda_k=\lambda_k}, \quad S_{Zr}^{(0)} = \left.\frac{\partial W}{\partial \Lambda_1}\right|_{\Lambda_k=\lambda_k}\cos\beta, \quad S_{Zz}^{(0)} = \left.\frac{\partial W}{\partial \Lambda_1}\right|_{\Lambda_k=\lambda_k}\sin\beta. \tag{3.70}$$

It follows from (3.64) that

$$T_1^{(0)} = \left.\frac{\partial W}{\partial \Lambda_1}\right|_{\Lambda_k=\lambda_k}, \quad T_2^{(0)} = \left.\frac{\partial W}{\partial \Lambda_2}\right|_{\Lambda_k=\lambda_k}. \tag{3.71}$$

4. Wave propagation in a non-linear hyperelastic membrane tube

4.1. Formulation of the problem

In this section we study wave propagation in a circular cylindrical non-linear elastic tube when the tube is subjected to dynamic extension. We consider a circular cylindrical membrane of homogeneous incompressible isotropic non-linearly elastic material. In its (undeformed, stress free) natural configuration the membrane tube is defined by (2.13). We suppose that the ends of the tube in the natural configuration are attached to rigid rings or discs of radius R_{i}. We also suppose that the end of the tube at $Z = L$ is kept fixed and the end originally at $Z = 0$ is displaced in the negative direction.

The same problem has been studied by Tait and Zhong [5]. They have considered a numerical technique based on the method of characteristics. The present discussion of the problem is set in a framework which differs in several aspects from that studied by Tait and Zhong [5]. First we derive the governing equations using an alternative, but equivalent, approach to that used in [5]. In particular, we employ the membrane theory derived in the previous section by using an asymptotic expansion technique, whereas they formulate membrane theory following the direct approach of Green et al. [8] and regard the membrane as an elastic surface in three-dimensional space. The equivalence of the membrane theories, obtained by using the direct theory and by using an asymptotic expansion technique, is not the subject of the present study. For a discussion about the equivalence of these membrane theories we refer to [3]. Second, for the numerical solution of the governing equations we employ a modified second-order Godunov method. We also discuss a possible existence of shock waves numerically and how the results of Tait and Zhong [5] are related to those obtained in the present study. For convenience we drop the superscript 0 which is used to denote the zeroth-order quantities.

In the absence of the normal and tangential surface tractions (that is, $P_{\mathrm{n}} = P_{\mathrm{t}} \equiv 0$), the equations of motion given in (3.65) and (3.66) take the following form:

$$\frac{\partial}{\partial Z}(T_1 \cos\beta) - T_2 = \frac{\partial^2 g}{\partial t^2}, \tag{4.1}$$

$$\frac{\partial}{\partial Z}(T_1 \sin\beta) = \frac{\partial^2 f}{\partial t^2}. \tag{4.2}$$

If we set

$$u = \frac{\partial g}{\partial t}, \quad v = \frac{\partial f}{\partial t}, \tag{4.3}$$

we can write the following compatibility equations:

$$\frac{\partial(\lambda_1 \cos\beta)}{\partial t} = \frac{\partial u}{\partial Z}, \quad \frac{\partial(\lambda_1 \sin\beta)}{\partial t} = \frac{\partial v}{\partial Z}, \quad \frac{\partial \lambda_2}{\partial t} = u. \tag{4.4}$$

If these compatibility equations are combined with the equations of motion given by (4.1) and (4.2), the governing equations may be rewritten in the vector form

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{H}(\mathbf{Q})}{\partial Z} + \mathbf{B}(\mathbf{Q}) = \mathbf{0}, \tag{4.5}$$

where

$$\mathbf{Q} = \{\lambda_1 \cos\beta, \lambda_1 \sin\beta, \lambda_2, u, v\}^{\mathrm{T}},$$
$$\mathbf{H} = -\{u, v, 0, T_1 \cos\beta, T_1 \sin\beta\}^{\mathrm{T}},$$
$$\mathbf{B} = -\{0, 0, u, -T_2, 0\}^{\mathrm{T}}. \tag{4.6}$$

The above system can be written in the form

$$\frac{\partial \mathbf{Q}}{\partial t} + \mathbf{A}(\mathbf{Q})\frac{\partial \mathbf{Q}}{\partial Z} + \mathbf{B} = \mathbf{0}, \tag{4.7}$$

where $\mathbf{A}(\mathbf{Q}) \equiv \partial \mathbf{H}(\mathbf{Q})/\partial \mathbf{Q}$ is the 5×5 Jacobian matrix with the non-zero entries given by

$$A_{14} = -1, \quad A_{25} = -1, \quad A_{41} = -\frac{\partial T_1}{\partial \lambda_1}\cos^2\beta - \frac{T_1}{\lambda_1}\sin^2\beta,$$

$$A_{42} = A_{51} = -\left(\frac{T_1}{\lambda_1} - \frac{\partial T_1}{\partial \lambda_1}\right)\sin\beta\cos\beta, \quad A_{43} = -\frac{\partial T_1}{\partial \lambda_2}\cos\beta,$$

$$A_{52} = -\frac{\partial T_1}{\partial \lambda_1}\sin^2\beta - \frac{T_1}{\lambda_1}\cos^2\beta, \quad A_{53} = -\frac{\partial T_1}{\partial \lambda_2}\sin\beta. \tag{4.8}$$

The eigenvalues of **A** are $\Delta_0 = 0$, $\Delta_{\pm 1} = \pm C_{\mathrm{L}}$, $\Delta_{\pm 2} = \pm C_{\mathrm{T}}$, where C_{L} and C_{T} are given by

$$C_{\mathrm{L}}^2 = \frac{\partial T_1}{\partial \lambda_1}, \quad C_{\mathrm{T}}^2 = \frac{T_1}{\lambda_1} \tag{4.9}$$

and represent the speeds of an essentially longitudinal wave and an essentially transverse wave, respectively. We require that the wave speeds are real.

To proceed with our analysis further, the boundary and initial conditions for the problem can be given as follows. Since the initial configuration is the reference configuration

$$g(Z,0) = 1, \quad f(Z,0) = Z, \tag{4.10}$$

one can write the initial conditions in the form

$$\lambda_1(Z,0) = \lambda_2(Z,0) = 1, \quad \beta(Z,0) = \pi/2, \quad u(Z,0) = v(Z,0) = 0, \tag{4.11}$$

for $0 \leqslant Z \leqslant L$. The boundary conditions for the present problem are

$$\lambda_2(0,t) = 1, \quad u(0,t) = 0, \quad v(0,t) = -v_0, \quad \lambda_1(L,t) = 1, \quad \beta(L,t) = \pi/2, \quad t > 0. \tag{4.12}$$

Here we assume that the waves do not arrive in the end $Z = L$ over the time interval considered.

Up to this point the strain-energy function W has remained arbitrary. It is not clear whether the wave speeds are real. As in [5], in order to obtain precise results we consider a membrane tube composed of the Mooney–Rivlin material for which the strain-energy function is given by

$$W(I_1, I_2) = \frac{\mu}{2}[\gamma(I_1 - 3) + (1-\gamma)(I_2 - 3)], \tag{4.13}$$

where the constants μ and γ satisfy $\mu > 0$ and $0 \leqslant \gamma \leqslant 1$, respectively. When $\gamma = 1$ in (4.13), one recovers the strain-energy function of the neo-Hookean material which has also received much attention in the literature. If we rewrite the above strain-energy function (4.13) in terms of the principal stretches Λ_1, Λ_2 by using (2.2) and substitute the resulting equation into (3.71), we obtain the Biot principal stresses, T_1 and T_2, in the form

$$T_1 = [\gamma + (1-\gamma)\lambda_2^2](\lambda_1 - \lambda_1^{-3}\lambda_2^{-2}), \quad T_2 = [\gamma + (1-\gamma)\lambda_1^2](\lambda_2 - \lambda_1^{-2}\lambda_2^{-3}). \tag{4.14}$$

It follows from (4.9) that

$$C_{\mathrm{L}}^2 = [\gamma + (1-\gamma)\lambda_2^2](1 + 3\lambda_1^{-4}\lambda_2^{-2}), \quad C_{\mathrm{T}}^2 = [\gamma + (1-\gamma)\lambda_2^2](1 - \lambda_1^{-4}\lambda_2^{-2}). \tag{4.15}$$

It is seen that $(4.15)_1$ implies $C_{\mathrm{L}}^2 > 0$ and that $(4.15)_2$ implies $C_{\mathrm{T}}^2 > 0$ provided $\lambda_1^2\lambda_2 > 1$. There exist four waves propagating in the tube and two of them propagate in the positive direction along the tube whereas the remaining two propagate in the negative direction. Since

$$C_{\mathrm{L}}^2 - C_{\mathrm{T}}^2 = 4[\gamma + (1-\gamma)\lambda_2^2)]\lambda_1^{-4}\lambda_2^{-4} > 0, \tag{4.16}$$

we have $C_{\mathrm{L}}^2 > C_{\mathrm{T}}^2 > 0$ whenever $\lambda_1^2\lambda_2 > 1$. Therefore, the eigenvalues of **A** are all real and distinct whenever $\lambda_1^2\lambda_2 > 1$. That is, system (4.5) is strictly hyperbolic and there is no degeneracy.

4.2. Numerical results

Since the high non-linearity of the equations and the presence of the source terms rule out analytical solutions, the initial and boundary-value problem defined by (4.5), (4.6), (4.11) and (4.12) is solved numerically by using a modified second-order Godunov method. The basic idea of Godunov-type methods is to compose the global solution by the exact or approximate solutions of local Riemann problems. The second-order method replaces the piecewise constant representation of the solution in the first-order method by a piecewise linear representation. Algorithms of this type were first introduced by van Leer in a series of papers [9]. For a more complete account of the numerical schemes of hyperbolic systems we refer to the recent books by Leveque [10], Godlewski and Raviart [11] and Kröner [12]. For an application of the numerical method to non-linear elasticity we refer to [13].

To apply the numerical method the computational domain $[0, L]$ is divided into M subintervals with equal length ΔZ where $\Delta Z = L/M$. Similarly, if the solution is computed up to time $t = t_F$, the time interval $[0, t_F]$ is divided into N subintervals with equal length Δt where $\Delta t = t_F/N$. We first compare the numerical results obtained for two different grids: a coarse one and a finer one. In these experiments we apply the numerical scheme for the parameter values given by $L = 2$, $v_0 = 1$, $\gamma = 0.3$, $t_F = 1$. In the case of the coarse grid we apply the numerical scheme with $M = 2000$ and $N = 4000$ which correspond to $\Delta Z = 10^{-3}$ and $\Delta t = 0.25 \times 10^{-3}$, respectively. In the case of the fine grid we apply the numerical scheme with $M = 10\,000$ and $N = 20\,000$ which correspond to $\Delta Z = 2 \times 10^{-4}$ and $\Delta t = 0.5 \times 10^{-4}$, respectively. Since $\max C_L \leqslant 2$ for the present problem, the well-known CFL (Courant, Friedrichs and Lewy) condition $\max C_L \leqslant \Delta x/2\Delta t$, which gives a restriction on Δt by using the largest wave speed, is satisfied in both cases. We observe that the numerical results obtained using the fine grid are not distinguishable from those obtained using the coarse grid. Therefore we present the numerical results obtained using the coarse grid in the remaining part of this section.

Figs. 1 and 2 show the values of λ_1 and λ_2 as functions of Z at various times. Recall that we take the undeformed configuration as the initial configuration. Therefore the membrane is undisturbed ahead of the fastest wave. The leading edge of the fastest wave travels at speed $C_L|_{\lambda_1=\lambda_2=1} = 2$. An expansion wave of λ_1 joins the undisturbed region to the region where interaction between the waves takes place. A comparison of the numerical results corresponding to λ_1 and λ_2 shows good qualitative agreement between the numerical results in the present study and those given in [5]. The only difference between the present results and those given in [5] is the presence of a discontinuity in λ_1' at the rear of the expansion fan in Fig. 1. This situation indicates that the numerical method in [5] smoothes the discontinuity in λ_1'. Since a slightly

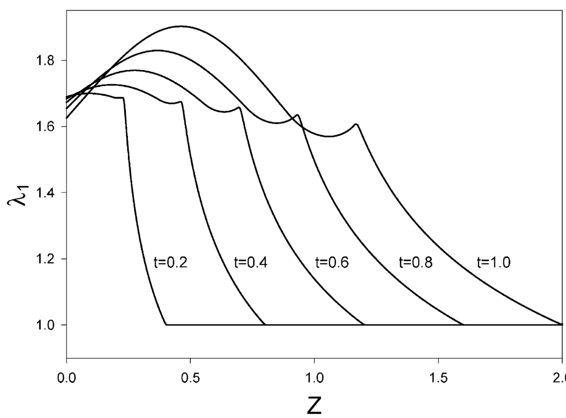

Fig. 1. λ_1 as a function of Z at various times. $\Delta Z = 10^{-3}$, $\Delta t = 0.25 \times 10^{-3}$, $L = 2$, $v_0 = 1$, $\gamma = 0.3$.

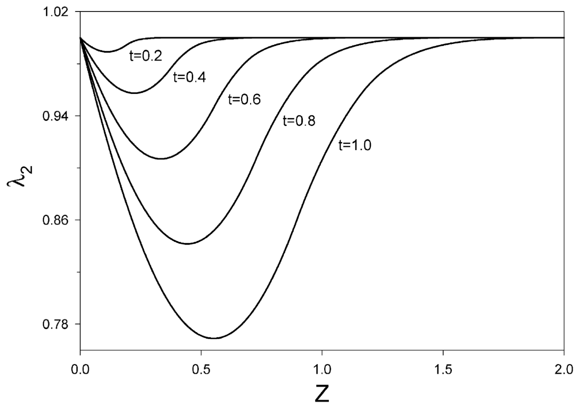

Fig. 2. λ_2 as a function of Z at various times. $\Delta Z = 10^{-3}$, $\Delta t = 0.25 \times 10^{-3}$, $L = 2$, $v_0 = 1$, $\gamma = 0.3$.

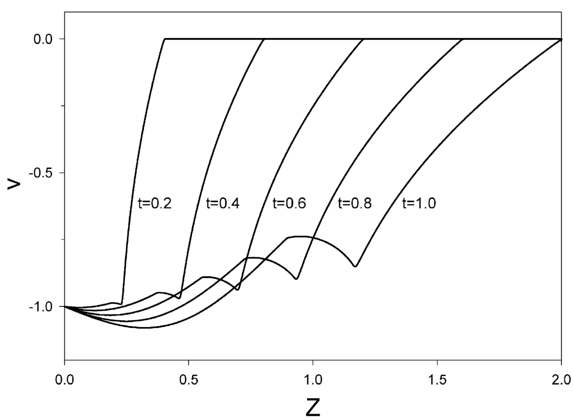

Fig. 3. v as a function of Z at various times. $\Delta Z = 10^{-3}$, $\Delta t = 0.25 \times 10^{-3}$, $L = 2$, $v_0 = 1$, $\gamma = 0.3$.

different non-dimensionalization is used in [5], a conversation factor of 2 is necessary for the horizontal axis.

Figs. 3 and 4 show the values of v and u as functions of Z at various times. Note that the governing equations cannot be separated into two uncoupled systems which govern the propagation of a longitudinal wave (associated with f' and v) and a transverse wave (associated with g' and u), respectively. It follows from (4.15), (3.50) and (3.29) that the wave speeds C_L and C_T are functions of f' and g' so that they are also coupled. Therefore a pure longitudinal or transverse wave cannot exist and interaction takes place. The results shown in Figs. 3 and 4 indicate that both essentially longitudinal wave and essentially transverse wave are acceleration waves and exhibit no discontinuities of v and u. It is evident from the figures that there are two jumps in v' and one in u' and that these discontinuities propagate at different speeds. Whereas one of the discontinuities in v' and the u' discontinuity propagate at the same speed, the other discontinuity in v' propagates faster than the u' discontinuity. These results indicate that the wave with speed C_L is an essentially v wave whereas the wave with speed C_T is an essentially u wave. The v wave exhibits a perturbation as it passes through the u' discontinuity and the slow discontinuity in v' results from the discontinuity in u'. This is a direct consequence of the coupling of the waves.

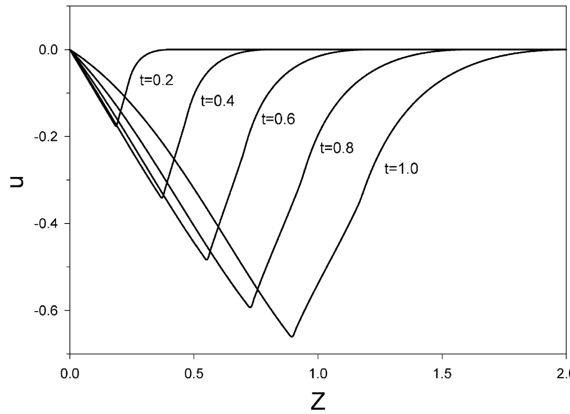

Fig. 4. u as a function of Z at various times. $\Delta Z = 10^{-3}$, $\Delta t = 0.25 \times 10^{-3}$, $L = 2$, $v_0 = 1$, $\gamma = 0.3$.

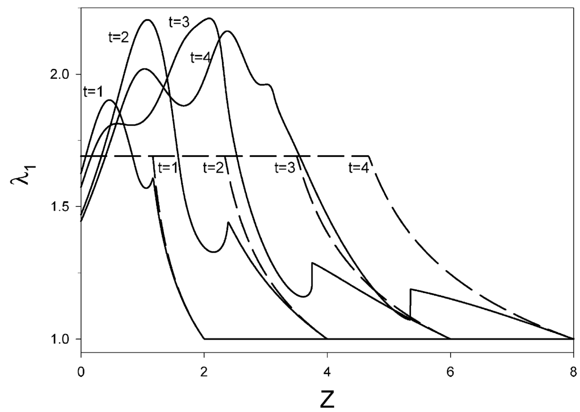

Fig. 5. λ_1 as a function of Z at various times. The dashed lines represent the simplified solution obtained from the limiting case. $\Delta Z = 10^{-3}$, $\Delta t = 0.25 \times 10^{-3}$, $L = 8$, $v_0 = 1$, $\gamma = 0.3$.

We now ask whether the discontinuities in λ_1' and v' are transformed into the discontinuities in λ_1 and v, respectively, over large time and space intervals. In order to investigate numerically whether shocks will eventually form we repeat the numerical experiment for greater values of t_F. In these experiments we also increase the value of L to prevent that the reflected wave modifies the behavior. At $t = 4.145$ the condition $\lambda_1^2 \lambda_2 > 1$ is violated and so the computation is halted. In Figs. 5 and 6 we show the values of λ_1 and v as functions of Z at various times for $L=8$ and $t_F=4$. The solid line in each figure indicates the numerical results obtained by solving the exact equations as in the previous figures. However, the dashed line in Figs. 5 and 6 indicates the exact solutions of the approximate equations which will be derived in the following paragraph. The results shown in Figs. 5 and 6 point out that two acceleration waves begin to form a shock wave as time increases. Although a formal proof of the existence of shock waves is not presented here, the above numerical results strongly indicate that a shock forms. For a discussion of the transition from acceleration wave to shock wave in the case of a pure longitudinal wave we refer to [14].

A limiting case has been considered by Tait and Zhong [5] in which the governing equations reduce to those for a stretched string. The closed form solutions obtained for the approximate set of equations have

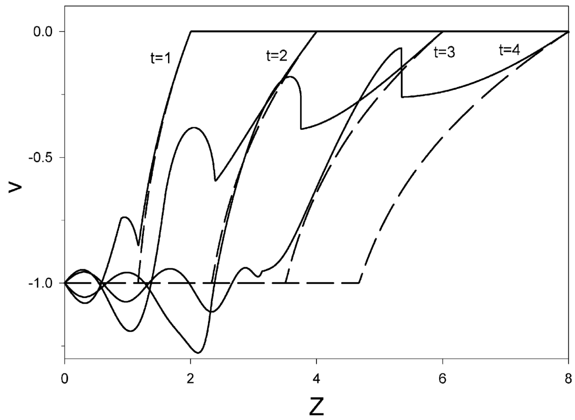

Fig. 6. v as a function of Z at various times. The dashed lines represent the simplified solution obtained from the limiting case. $\Delta Z = 10^{-3}$, $\Delta t = 0.25 \times 10^{-3}$, $L = 8$, $v_0 = 1$, $\gamma = 0.3$.

been used as a check on the numerical results presented for the full set of equations. We now follow a similar approach and use the exact solutions obtained for the limiting case as a check on the numerical results presented in Figs. 5 and 6. Specifically we deal with the limiting case of the loading problem when axial displacements are much larger than radial displacements. In other words, for the moment, we assume that a pure longitudinal wave can propagate without a coupled transverse wave. Then in the motion we approximate g, λ_2, u, g', λ_1 and β by $g = 1$, $\lambda_2 = 1$, $u = 0$, $g' = 0$, $\lambda_1 = f'$ and $\beta = \pi/2$, respectively. Then the system (4.5) reduces to

$$\frac{\partial}{\partial t}\begin{pmatrix}\lambda_1\\ v\end{pmatrix} + \begin{pmatrix}0 & -1\\ -c^2 & 0\end{pmatrix}\frac{\partial}{\partial Z}\begin{pmatrix}\lambda_1\\ v\end{pmatrix} = \mathbf{0}, \tag{4.17}$$

where

$$c^2 = \frac{\mathrm{d}T_1(\lambda_1)}{\mathrm{d}\lambda_1}, \quad T_1 = \frac{\mathrm{d}W(\lambda_1)}{\mathrm{d}\lambda_1}, \tag{4.18}$$

and $\pm c$ are the wave speeds associated with pure stretching waves. System (4.17) is identical to that given in equation (5.1) of [5]. As in [5], we consider the initial conditions

$$v(Z,0) = 0, \quad \lambda_1(Z,0) = 1 \tag{4.19}$$

for $0 \leqslant Z \leqslant L$ and the boundary conditions

$$v(0,t) = -v_0, \quad \lambda_1(L,t) = 1, \quad t > 0. \tag{4.20}$$

The explicit solution of the problem defined by (4.17), (4.19) and (4.20) has been given by Tait and Zhong [6]. The solution consists of two constant regions joined by a simple expansion fan and takes the following form in our notation:

(i) If $0 \leqslant Z \leqslant c^* t$ where $c^* = c(\lambda_1^*)$ and λ_1^* is determined by the equation

$$v_0 = \int_1^{\lambda_1^*} c(\lambda_1)\,\mathrm{d}\lambda_1, \tag{4.21}$$

the solution is given by

$$\lambda_1 = \lambda_1^*, \quad v = v_0. \tag{4.22}$$

(ii) If $c^* t \leqslant Z \leqslant 2t$, the solution is given by

$$\lambda_1^4 = \frac{3}{(Z/t)^2 - 1}, \quad v = -\int_1^{\lambda_1} c(\tau)\,\mathrm{d}\tau. \tag{4.23}$$

(iii) If $2t \leqslant Z \leqslant L$, the solution is given by

$$\lambda_1 = 1, \quad v = 0. \tag{4.24}$$

Further details on this solution may be found in [6]. In Figs. 5 and 6 the analytical solutions given by (4.22)–(4.24) are compared with the numerical solutions of the full set of equations. The dashed line in the figures indicates the simplified solution obtained from limiting case. In the limiting case the longitudinal wave is an acceleration wave and exhibits no discontinuities of λ_1 or v. As Z increases the numerical solution of the full equations approaches the exact solution of the limiting case. These results indicate that qualitative differences between the results for the original problem and the limiting case result from the coupled transverse wave.

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