Blow-up and global existence for a general class of nonlocal nonlinear coupled wave equations

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Abstract

We study the initial-value problem for a general class of nonlinear nonlocal coupled wave equations. The problem involves convolution operators with kernel functions whose Fourier transforms are nonnegative. Some well-known examples of nonlinear wave equations, such as coupled Boussinesq-type equations arising in elasticity and in quasi-continuum approximation of dense lattices, follow from the present model for suitable choices of the kernel functions. We establish local existence and sufficient conditions for finite time blow-up and as well as global existence of solutions of the problem.

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1. Introduction

In this article we focus on blow-up and global existence of solutions to the nonlocal nonlinear Cauchy problem

$$u_{1tt} = (\beta_1 * (u_1 + g_1(u_1, u_2)))_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0 \tag{1.1}$$
$$u_{2tt} = (\beta_2 * (u_2 + g_2(u_1, u_2)))_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0 \tag{1.2}$$
$$u_1(x,0) = \varphi_1(x), \quad u_{1t}(x,0) = \psi_1(x) \tag{1.3}$$
$$u_2(x,0) = \varphi_2(x), \quad u_{2t}(x,0) = \psi_2(x). \tag{1.4}$$

Here $u_i = u_i(x,t)$ $(i = 1, 2)$, the subscripts x, t denote partial derivatives, the symbol $*$ denotes convolution in the spatial domain

$$\beta * v = \int_{\mathbb{R}} \beta(x-y) v(y) \mathrm{d}y.$$

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We assume that the nonlinear functions $g_i(u_1, u_2)$ $(i = 1, 2)$ satisfy the exactness condition

$$\frac{\partial g_1}{\partial u_2} = \frac{\partial g_2}{\partial u_1} \tag{1.5}$$

or equivalently there exists a function $G(u_1, u_2)$ satisfying

$$g_i = \frac{\partial G}{\partial u_i} \qquad (i = 1, 2). \tag{1.6}$$

We assume that the kernel functions $\beta_i(x)$ are integrable and their Fourier transforms $\widehat{\beta}_i(\xi)$ satisfy

$$0 \le \widehat{\beta}_i(\xi) \le C_i(1 + \xi^2)^{-r_i/2} \text{ for all } \xi \qquad (i = 1, 2) \tag{1.7}$$

for some constants $C_i > 0$. Here the exponents r_1, r_2 are not necessarily integers.

Equations (1.1)-(1.2) may be viewed as a natural generalization of the single equation arising in one-dimensional nonlocal elasticity [1] to a coupled system of two nonlocal nonlinear equations. As a special case, consider $g_i(u_1, u_2) = u_i W'(u_1^2 + u_2^2)$ $(i = 1, 2)$ where W is a function of $u_1^2 + u_2^2$ alone and the symbol $'$ denotes the derivative. Then, (1.1)-(1.2) may be thought of the system governing the one-dimensional propagation of two "pure" transverse nonlinear waves in a nonlocal elastic isotropic homogeneous medium [2]. Note that this choice of g_1 and g_2 will satisfy the exactness condition (1.5) with $G(u_1, u_2) = \frac{1}{2}W(u_1^2 + u_2^2)$. From the modelling point of view we want to remark that, in general, the system will also contain a third equation characterizing the propagation of a longitudinal wave. Nevertheless, with some further restrictions imposed on the form of W, one may get transverse waves without a coupled longitudinal wave [3]. We also want to note that, in the general case, the exactness condition (1.5) is necessary in order to obtain the conservation law of Lemma 3.2.

For suitable choices of the kernel functions, the system (1.1)-(1.2) reduces to some well-known coupled systems of nonlinear wave equations. To illustrate this we consider the exponential kernel $\beta_1(x) = \beta_2(x) = \frac{1}{2}e^{-|x|}$ which is the Green's function for the operator $1 - D_x^2$ where D_x stands for the partial derivative with respect to x. Then, applying the operator $1 - D_x^2$ to both sides of equations (1.1)-(1.2) yields the coupled improved Boussinesq equations

$$u_{1tt} - u_{1xx} - u_{1xxtt} = (g_1(u_1, u_2))_{xx} \tag{1.8}$$

$$u_{2tt} - u_{2xx} - u_{2xxtt} = (g_2(u_1, u_2))_{xx}. \tag{1.9}$$

Similarly, if the kernels $\beta_1(x)$ and $\beta_2(x)$ are chosen as the Green's function for the fourth-order operator $1 - aD_x^2 + bD_x^4$ with positive constants a, b, then (1.1)-(1.2) reduces to the coupled higher-order Boussinesq system

$$u_{1tt} - u_{1xx} - au_{1xxtt} + bu_{1xxxxtt} = (g_1(u_1, u_2))_{xx} \tag{1.10}$$

$$u_{2tt} - u_{2xx} - au_{2xxtt} + bu_{2xxxxtt} = (g_2(u_1, u_2))_{xx}. \tag{1.11}$$

These examples make it obvious that choosing the kernels $\beta_i(x)$ in (1.1)-(1.2) as the Green's functions of constant coefficient linear differential operators in x will yield similar coupled systems describing the bi-directional propagation of nonlinear waves in dispersive media. Different examples of the kernel functions used in the literature can be found in [1] where such kernels will give not only differential equations but also integro-differential equations or difference-differential equations. For a survey of Korteweg-de Vries type nonlinear nonlocal equations of hydrodynamic relevance we refer to [4].

The coupled improved Boussinesq system (1.8)-(1.9) has been derived to describe bi-directional wave propagation in various contexts, for instance, in a Toda lattice model with a transversal degree of freedom [5], in a two-layered lattice model [6] and in a diatomic lattice [7]. For a discussion of the classical Boussinesq system we refer to [8, 9]. The Cauchy problem for (1.8)-(1.9) has been studied in [10] and recently in [11] where both assume the exactness condition (1.5). They have established the conditions for the global existence and finite-time blow-up of solutions in Sobolev spaces $H^s \times H^s$ for $s > 1/2$.

The single component form of equations (1.10)-(1.11) arises as a model for a dense chain of particles with elastic couplings [12], for water waves with surface tension [13] and for longitudinal waves in a nonlocal nonlinear elastic medium [2]. We have proved in [2] that the Cauchy problem for the single component form of (1.10)-(1.11) is globally well-posed in Sobolev spaces H^s for $s > 1/2$ under certain conditions on nonlinear term and initial data. To the best of our knowledge, the questions of global well-posedness and finite-time blow-up of solutions for the coupled higher-order Boussinesq system (1.10)-(1.11) are open problems. In this article we shall resolve these problems by considering a closely related, but somewhat more general, problem defined by (1.1)-(1.4)

In Section 2 we present a local existence theory of the Cauchy problem (1.1)-(1.4) for the case of general kernels with $r_1, r_2 \geq 2$ and initial data in suitable Sobolev spaces. In Section 3 we prove the energy identity and in Section 4 we discuss finite time blow-up of solutions of the initial-value problem. Finally, in Section 5 we prove two separate results on global existence of solutions of (1.1)-(1.4) for two different classes of kernel functions.

In what follows $H^s = H^s(\mathbb{R})$ will denote the L^2 Sobolev space on $\mathbb{R}$. For the H^s norm we use the Fourier transform representation $\|u\|_s^2 = \int_{\mathbb{R}}(1+\xi^2)^s|\hat{u}(\xi)|^2 \mathrm{d}\xi$. We use $\|u\|_\infty$, $\|u\|$ and $\langle u, v\rangle$ to denote the L^∞ and L^2 norms and the inner product in L^2, respectively.

2. Local Well Posedness

To shorten the notation we write $f_i(u_1, u_2) = u_i + g_i(u_1, u_2)$ $(i = 1, 2)$. Note that

$$f_i = \frac{\partial F}{\partial u_i} \qquad (i = 1, 2) \tag{2.1}$$

where $F(u_1, u_2) = \frac{1}{2}(u_1^2 + u_2^2) + G(u_1, u_2)$.

For a vector function $U = (u_1, u_2)$ we define the norms $\|U\|_s = \|u_1\|_s + \|u_2\|_s$ and $\|U\|_\infty = \|u_1\|_\infty + \|u_2\|_\infty$. We first need vector-valued versions of Lemma 3.1 and Lemma 3.2 in [1] (see also [11, 14, 15]), which concern the behavior of the nonlinear terms:

Lemma 2.1. *Let $s \geq 0$, $h \in C^{[s]+1}(\mathbb{R}^2)$ with $h(0) = 0$. Then for any $U = (u_1, u_2) \in (H^s \cap L^\infty)^2$, we have $h(U) \in H^s \cap L^\infty$. Moreover there is some constant $A(M)$ depending on M such that for all $U \in (H^s \cap L^\infty)^2$ with $\|U\|_\infty \leq M$*

$$\|h(U)\|_s \leq A(M)\|U\|_s \ .$$

Lemma 2.2. *Let $s \geq 0$, $h \in C^{[s]+1}(\mathbb{R}^2)$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $U, V \in (H^s \cap L^\infty)^2$ with $\|U\|_\infty \leq M$, $\|V\|_\infty \leq M$ and $\|U\|_s \leq M$, $\|V\|_s \leq M$ we have*

$$\|h(U) - h(V)\|_s \leq B(M)\|U - V\|_s \quad \text{and} \quad \|h(U) - h(V)\|_\infty \leq B(M)\|U - V\|_\infty .$$

The Sobolev embedding theorem implies that $H^s \subset L^\infty$ for $s > \frac{1}{2}$. Then the bounds on L^∞ norms in Lemma 2.2 appear unnecessary and we get:

Corollary 2.3. *Let $s > \frac{1}{2}$, $h \in C^{[s]+1}(\mathbb{R}^2)$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $U, V \in (H^s)^2$ with $\|U\|_s \leq M$, $\|V\|_s \leq M$ we have*

$$\|h(U) - h(V)\|_s \leq B(M)\|U - V\|_s .$$

Throughout this paper we assume that $f_1, f_2 \in C^\infty(\mathbb{R}^2)$ with $f_1(0) = f_2(0) = 0$. In the case of $f_1, f_2 \in C^{k+1}(\mathbb{R}^2)$, Lemma 2.1 and Lemma 2.2 will hold only for $s \leq k$. Thus all the results below will hold for $s \leq k$. Note that the functions g_1 and g_2 appearing in (1.1) and (1.2) will also satisfy the same assumptions as f_1 and f_2.

Theorem 2.4. *Let $s > 1/2$ and $r_1, r_2 \geq 2$. Then there is some $T > 0$ such that the Cauchy problem (1.1)-(1.4) is well posed with solution u_1 and u_2 in $C^2([0,T], H^s)$ for initial data $\varphi_i, \psi_i \in H^s$ $(i = 1, 2)$.*

Proof. We convert the problem into an H^s valued system of ordinary differential equations

$$\begin{array}{ll} u_{1t} = v_1, & u_1(0) = \varphi_1, \\ u_{2t} = v_2, & u_2(0) = \varphi_2, \\ v_{1t} = \beta_1 * (f_1(u_1, u_2))_{xx}, & v_1(0) = \psi_1, \\ v_{2t} = \beta_2 * (f_2(u_1, u_2))_{xx}, & v_2(0) = \psi_2. \end{array}$$

In order to use the standard well-posedness result [16] for ordinary differential equations, it suffices to show that the right hand side is Lipschitz on H^s. Since $r_i \geq 2$ for $i = 1, 2$, we have

$$\left| -\xi^2 \widehat{\beta}_i(\xi) \right| \leq C_i \xi^2 (1 + \xi^2)^{-r_i/2} \leq C_i.$$

Then we get

$$\begin{aligned} \|\beta_i * w_{xx}\|_s &= \left\| (1 + \xi^2)^{s/2} \xi^2 \widehat{\beta}_i(\xi) \widehat{w}(\xi) \right\| \\ &\leq C_i \left\| (1 + \xi^2)^{s/2} \widehat{w}(\xi) \right\| = C_i \|w\|_s . \end{aligned} \tag{2.2}$$

This implies that $\beta_i * (.)_{xx}$ is a bounded linear map on H^s. Then it follows from Corollary 2.3 that $\beta_i * (f_i(u_1, u_2))_{xx}$ is locally Lipschitz on H^s for $s > \frac{1}{2}$. □

Remark 2.5. *In Theorem 2.4 we have not used neither the assumption $\widehat{\beta}(\xi) \geq 0$ nor the exactness condition (1.5); so in fact the local existence result holds for more general forms of kernel functions and nonlinear terms. Moreover, as in [1], for certain classes of kernel functions Theorem 2.4 can be extended to the case of $H^s \cap L^\infty$ for $0 \leq s \leq 1/2$.*

The solution in Theorem 2.4 can be extended to a maximal time interval of existence $[0, T_{\max})$ where finite $T_{\max}$ is characterized by the blow up condition

$$\limsup_{t\to T_{\max}^-}(\|U(t)\|_s + \|U_t(t)\|_s) = \infty,$$

where $U_t = (u_{1t}, u_{2t})$. Then the solution is global, i.e. $T_{\max} = \infty$ iff

$$\text{for any } T < \infty, \text{ we have } \limsup_{t\to T^-}(\|U(t)\|_s + \|U_t(t)\|_s) < \infty \ . \tag{2.3}$$

Lemma 2.6. *Let $s > 1/2$, $r_1, r_2 \geq 2$ and let U be the solution of the Cauchy problem (1.1)-(1.4). Then there is a global solution if and only if*

$$\textit{for any } \ T < \infty, \ \textit{we have } \limsup_{t\to T^-}\|U(t)\|_\infty < \infty \ . \tag{2.4}$$

Proof. We will show that the two conditions (2.3) and (2.4) are equivalent. First assume that (2.3) holds. By the Sobolev imbedding theorem, $\|U(t)\|_\infty \leq C\,\|U(t)\|_s$ for $s > 1/2$ so (2.4) holds. Conversely, assume that the solution exists for $t \in [0, T)$. Then $M = \limsup_{t\to T^-}\|U(t)\|_\infty$ is finite and $\|U(t)\|_\infty \leq M$ for all $0 \leq t \leq T$. If we integrate (1.1)-(1.2) twice and compute the resulting double integral as an iterated integral, we get, for $i = 1, 2$,

$$u_i(x,t) = \varphi_i(x) + t\psi_i(x) + \int_0^t (t-\tau)(\beta_i * f_i(u_1, u_2))_{xx}(x,\tau)d\tau \tag{2.5}$$

$$u_{it}(x,t) = \psi_i(x) + \int_0^t (\beta_i * f_i(u_1, u_2))_{xx}(x,\tau)d\tau \ . \tag{2.6}$$

So, for all $t \in [0, T)$ and $i = 1, 2$

$$\begin{aligned}
\|u_i(t)\|_s &\leq \|\varphi_i\|_s + T\,\|\psi_i\|_s + T\int_0^t \|(\beta_i * f_i(u_1,u_2))_{xx}(\tau)\|_s\,d\tau, \\
\|u_{it}(t)\|_s &\leq \|\psi_i\|_s + \int_0^t \|(\beta_i * f_i(u_1,u_2))_{xx}(\tau)\|_s\,d\tau \ .
\end{aligned}$$

Note that $\|(\beta_i * f_i(u_1,u_2))_{xx}(\tau)\|_s \leq C_i\|f_i(u_1,u_2)(\tau)\|_s \leq C_iA_i(M)\|u_i(\tau)\|_s$ where the first inequality follows from (2.2) and the second from Lemma 2.1. Adding the four inequalities we get

$$\begin{aligned}
\|U(t)\|_s + \|U_t(t)\|_s &\leq \|\varphi_1\|_s + \|\varphi_2\|_s + (T+1)(\|\psi_1\|_s + \|\psi_2\|_s) \\
&+ (T+1)CA(M)\int_0^t \|U(\tau)\|_s\,d\tau,
\end{aligned}$$

where $C = \max(C_1, C_2)$ and $A(M) = \max(A_1(M), A_2(M))$. Gronwall's Lemma implies

$$\|U(t)\|_s + \|U_t(t)\|_s \leq [\|\varphi_1\|_s + \|\varphi_2\|_s + (T+1)(\|\psi_1\|_s + \|\psi_2\|_s)]e^{(T+1)CA(M)T}$$

for all $t \in [0, T)$ and consequently

$$\limsup_{t\to T^-}\ (\|U(t)\|_s + \|U_t(t)\|_s) < \infty \ .$$

□

3. Conservation of Energy

In the rest of the study we will assume that $\widehat{\beta}_i(\xi)$ has only isolated zeros for $i = 1, 2$. Let P_i be operator defined by $P_i w = \mathcal{F}^{-1}(|\xi|^{-1} (\widehat{\beta}_i(\xi))^{-1/2} \widehat{w}(\xi))$ with the inverse Fourier transform $\mathcal{F}^{-1}$. Note that although P_i may fail to be a bounded operator, its inverse $P_i^{-1} : H^{s+\frac{r_i}{2}} \to H^s$ is bounded and one-to-one for $s \geq 0$. Then P_i is well defined with $domain(P_i) = range(P_i^{-1})$. Clearly, $P_i^{-2} w = -(\beta_i * w)_{xx} = -\beta_i * w_{xx}$.

Lemma 3.1. *Let $s > 1/2$ and $r_1, r_2 \geq 2$. Suppose the solution of the Cauchy problem (1.1)-(1.4) exists with u_1 and u_2 in $C^2([0,T), H^s \cap L^\infty)$ for some $s > 1/2$. If $P_1\psi_1, P_2\psi_2 \in L^2$, then $P_1 u_{1t}, P_2 u_{2t} \in C^1([0,T), L^2)$. If moreover $P_1\varphi_1, P_2\varphi_2 \in L^2$, then $P_1 u_1, P_2 u_2 \in C^2([0,T), L^2)$.*

Proof. It follows from (2.6) that for $i = 1, 2$

$$P_i u_{it}(x,t) = P_i \psi_i(x) - \int_0^t (P_i^{-1} f_i(u_1, u_2))(x, \tau) d\tau \ .$$

It is clear from Lemma 2.1 that $f_i(u_1, u_2) \in H^s$. Also, $P_i^{-1} w = \mathcal{F}^{-1}(|\xi| (\widehat{\beta}_i(\xi))^{1/2} \widehat{w}(\xi))$ thus $P_i^{-1}(f_i(u_1, u_2)) \in H^{s+\frac{r_i}{2}-1} \subset L^2$ and hence $P_i u_{it} \in L^2$. The continuity and differentiability of $P_i u_i$ in t follows from the integral representation above. With a similar approach (2.5) gives the second statement. □

Lemma 3.2. *Let $s > 1/2$ and $r_1, r_2 \geq 2$. Suppose that (u_1, u_2) satisfies (1.1)-(1.4) on some interval $[0, T)$. If $P_1\psi_1, P_2\psi_2 \in L^2$ and the function $G(\varphi_1, \varphi_2)$ defined by (1.6) belongs to L^1, then for any $t \in [0, T)$ the energy*

$$\begin{aligned} E(t) &= \|P_1 u_{1t}(t)\|^2 + \|P_2 u_{2t}(t)\|^2 + \|u_1(t)\|^2 + \|u_2(t)\|^2 + 2\int_{\mathbb{R}} G(u_1, u_2) dx \\ &= \|P_1 u_{1t}(t)\|^2 + \|P_2 u_{2t}(t)\|^2 + 2\int_{\mathbb{R}} F(u_1, u_2) dx \end{aligned}$$

is constant in $[0, T)$.

Proof. Lemma 3.1 says that $P_i u_{it}(t) \in L^2$ for $i = 1, 2$. Equations (1.1)-(1.2) become $P_i^2 u_{itt} + u_i + g_i(u_1, u_2) = 0$ $(i = 1, 2)$. Multiplying by $2u_{it}$, integrating in x, adding the two equalities and using Parseval's identity we obtain $\frac{dE}{dt} = 0$. □

4. Blow Up in Finite Time

The following lemma will be used in the sequel to prove blow up in finite time.

Lemma 4.1. *[17, 18] Suppose $\Phi(t)$, $t \geq 0$ is a positive, twice differentiable function satisfying $\Phi''\Phi - (1+\nu)(\Phi')^2 \geq 0$ where $\nu > 0$. If $\Phi(0) > 0$ and $\Phi'(0) > 0$, then $\Phi(t) \to \infty$ as $t \to t_1$ for some $t_1 \leq \Phi(0)/\nu\Phi'(0)$.*

Theorem 4.2. *Let $s > 1/2$ and $r_1, r_2 \geq 2$. Suppose that $P_1\varphi_1, P_2\varphi_2, P_1\psi_1, P_2\psi_2 \in L^2$ and $G(\varphi_1, \varphi_2) \in L^1$. If there is some $\nu > 0$ such that*

$$u_1 f_1(u_1, u_2) + u_2 f_2(u_1, u_2) \leq 2(1+2\nu) F(u_1, u_2),$$

and

$$E(0) = \|P_1\psi_1\|^2 + \|P_2\psi_2\|^2 + 2\int_{\mathbb{R}} F(\varphi_1, \varphi_2)dx < 0,$$

then the solution (u_1, u_2) *of the Cauchy problem (1.1)-(1.4) blows up in finite time.*

Proof. Let

$$\Phi(t) = \|P_1u_1(t)\|^2 + \|P_2u_2(t)\|^2 + b(t + t_0)^2$$

for some positive b and t_0 that will be specified later. Assume that the maximal time of existence of the solution of the Cauchy problem (1.1)-(1.4) is infinite. Then $P_1u_1(t), P_1u_{1t}(t), P_2u_2(t), P_2u_{2t}(t) \in L^2$ for all $t > 0$; thus $\Phi(t)$ must be finite for all t. However, we will show below that $\Phi(t)$ blows up in finite time.

We have

$$\begin{aligned}
&\Phi'(t) = 2\langle P_1u_1, P_1u_{1t}\rangle + 2\langle P_2u_2, P_2u_{2t}\rangle + 2b(t + t_0),\\
&\Phi''(t) = 2\|P_1u_{1t}\|^2 + 2\|P_2u_{2t}\|^2 + 2\langle P_1u_1, P_1u_{1tt}\rangle + 2\langle P_2u_2, P_2u_{2tt}\rangle + 2b\ .
\end{aligned}$$

Since

$$\langle P_iu_i, P_iu_{itt}\rangle = \left\langle u_i, P_i^2u_{itt}\right\rangle = -\langle u_i, f_i(u_1, u_2)\rangle\,, \qquad i = 1, 2,$$

and

$$\begin{aligned}
-\int [u_1f_1(u_1, u_2) + u_2f_2(u_1, u_2)]dx \quad &\geq \quad -2(1 + 2\nu)\int F(u_1, u_2)dx\\
&= \quad (1 + 2\nu)(\|P_1u_{1t}(t)\|^2 + \|P_2u_{2t}(t)\|^2 - E(0)),
\end{aligned}$$

we get

$$\begin{aligned}
\Phi''(t) \quad &\geq \quad 2\|P_1u_{1t}\|^2 + 2\|P_2u_{2t}\|^2 + 2b - 2(1 + 2\nu)(E(0) - \|P_1u_{1t}\|^2 - \|P_2u_{2t}\|^2)\\
&= \quad -2(1 + 2\nu)E(0) + 2b + 4(1 + \nu)(\|P_1u_{1t}\|^2 + \|P_2u_{2t}\|^2).
\end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
(\Phi'(t))^2 \quad &= \quad 4[\langle P_1u_1, P_1u_{1t}\rangle + \langle P_2u_2, P_2u_{2t}\rangle + b(t + t_0)]^2\\
&\leq \quad 4[\|P_1u_1\|\,\|P_1u_{1t}\| + \|P_2u_2\|\,\|P_2u_{2t}\| + b(t + t_0)]^2\ .
\end{aligned}$$

For the mixed terms we use the inequalities

$$2\|P_1u_1\|\,\|P_1u_{1t}\|\,\|P_2u_2\|\,\|P_2u_{2t}\| \leq \|P_1u_1\|^2\,\|P_2u_{2t}\|^2 + \|P_2u_2\|^2\,\|P_1u_{1t}\|^2$$

and

$$2\|P_iu_i\|\,\|P_iu_{it}\|\,(t + t_0) \leq \|P_iu_i\|^2 + \|P_iu_{it}\|^2\,(t + t_0)^2, \qquad i = 1, 2,$$

to obtain

$$(\Phi'(t))^2 \leq 4\Phi(t)(\|P_1u_{1t}\|^2 + \|P_2u_{2t}\|^2 + b)\ .$$

Therefore,

$$\begin{aligned}
&\Phi(t)\Phi''(t) - (1 + \nu)(\Phi'(t))^2\\
&\qquad\qquad \geq\ \Phi(t)[-2(1 + 2\nu)E(0) + 2b + 4(1 + \nu)(\|P_1u_{1t}\|^2 + \|P_2u_{2t}\|^2)]\\
&\qquad\qquad\qquad\quad - 4(1 + \nu)\Phi(t)(\|P_1u_{1t}\|^2 + \|P_2u_{2t}\|^2 + b)\\
&\qquad\qquad = -2(1 + 2\nu)(E(0) + b)\Phi(t)\ .
\end{aligned}$$

If we choose $b \le -E(0)$, then $\Phi(t)\Phi''(t) - (1+\nu)(\Phi'(t))^2 \ge 0$. Moreover

$$\Phi'(0) = 2\left\langle P_1\varphi_1, P_1\psi_1\right\rangle + 2\left\langle P_2\varphi_2, P_2\psi_2\right\rangle + 2bt_0 \ge 0$$

for sufficiently large t_0. According to Lemma 4.1, we observe that $\Phi(t)$ blows up in finite time. This contradicts with the assumption of the existence of a global solution. □

Remark 4.3. *The proof above implies that we may have blow-up even if* $E(0) > 0$. *In this case, all we need is to be able to choose* b *and* t_0 *so that* $\Phi(0) > 0$ *and* $\Phi'(0) > 0$. *To shorten the notation put*

$$A = \left\langle P_1\varphi_1, P_1\psi_1\right\rangle + \left\langle P_2\varphi_2, P_2\psi_2\right\rangle, \qquad B = \left\|P_1\varphi_1\right\|^2 + \left\|P_2\varphi_2\right\|^2.$$

When $E(0) > 0$, *by choosing* $b = -E(0)$ *we still get blow up if there is some* t_0 *so that initial data satisfies*

$$A - E(0)t_0 > 0, \qquad B - E(0)t_0^2 > 0.$$

When $A > 0$, *taking* $t_0 = 0$ *works. When* $A \le 0$, *then* t_0 *must be chosen negative. The two inequalities can be rewritten as*

$$E(0)^{-2}A^2 < t_0^2, \qquad t_0^2 < E(0)^{-1}B.$$

Such a t_0 *exists if and only if* $A^2 < E(0)B$. *Hence there is blow-up if the initial data satisfies*

$$\left(\left\langle P_1\varphi_1, P_1\psi_1\right\rangle + \left\langle P_2\varphi_2, P_2\psi_2\right\rangle\right)^2 < E(0)\left(\left\|P_1\varphi_1\right\|^2 + \left\|P_2\varphi_2\right\|^2\right).$$

5. Global Existence

Below we prove global existence of solutions of (1.1)-(1.4) for two different classes of kernel functions. We note that the kernel functions corresponding to these two particular cases belong to the classes of kernel functions mentioned in Remark 2.5. Thus, in the cases below, the local existence result of Theorem 2.4, and hence Theorems 5.1 and 5.2 can be extended to $s \ge 0$ for initial data in $H^s \cap L^\infty$.

5.1. Sufficiently Smooth Kernels: $r_1, r_2 > 3$

We will now consider kernels β_i $(i = 1, 2)$ that satisfy the estimate $0 \le \widehat{\beta_i}(\xi) \le C_i(1+\xi^2)^{-r_i/2}$ with $r_i > 3$. Typically if β_i belongs to the Sobolev space $W^{3,1}(\mathbb{R})$ (i.e. β_i and its derivatives up to third-order are in L^1); then we would get the estimate with $r_i = 3$; hence we consider kernels that are slightly smoother than those in $W^{3,1}(\mathbb{R})$.

Theorem 5.1. *Let* $s > 1/2$, $r_1, r_2 > 3$. *Let* $\varphi_i, \psi_i \in H^s$, $P_i\psi_i \in L^2$ $(i = 1, 2)$ *and* $G(\varphi_1, \varphi_2) \in L^1$. *If there is some* $k > 0$ *so that* $G(a, b) \ge -k(a^2 + b^2)$ *for all* $a, b \in \mathbb{R}$, *then the Cauchy problem (1.1)-(1.4) has a global solution with* u_1 *and* u_2 *in* $C^2([0,\infty), H^s)$.

Proof. Since $r_1, r_2 > 3$, by Theorem 2.4 we have local existence. The hypothesis implies that $E(0) < \infty$. Assume that u_1, u_2 exist on $[0, T)$ for some $T > 0$. Since $G(u_1, u_2) \ge -k(u_1^2 + u_2^2)$, we get for all $t \in [0, T)$

$$\left\|P_1u_{1t}(t)\right\|^2 + \left\|P_2u_{2t}(t)\right\|^2 \le E(0) + (2k-1)(\left\|u_1(t)\right\|^2 + \left\|u_2(t)\right\|^2). \qquad (5.1)$$

Noting that $\widehat{\beta}_i(\xi) \le C_i(1+\xi^2)^{-r_i/2}$ for $i = 1,2$; we have

$$\begin{aligned}
\|P_1 u_{1t}(t)\|^2 = \left\|\widehat{P_1 u_{1t}}(t)\right\|^2 &= \int \xi^{-2}(\widehat{\beta}_1(\xi))^{-1}(\widehat{u}_{1t}(\xi,t))^2 d\xi \\
&\ge C_1^{-1}\int \xi^{-2}(1+\xi^2)^{r_1/2}(\widehat{u}_{1t}(\xi,t))^2 d\xi \\
&\ge C_1^{-1}\int (1+\xi^2)^{(r_1-2)/2}(\widehat{u}_{1t}(\xi,t))^2 d\xi \\
&= C_1^{-1}\|u_{1t}(t)\|_{\rho_1}^2 \ , \qquad (5.2)
\end{aligned}$$

and similarly,

$$\|P_2 u_{2t}(t)\|^2 \ge C_2^{-1}\|u_{2t}(t)\|_{\rho_2}^2 \qquad (5.3)$$

where $\rho_i = \frac{r_i}{2} - 1, \ \ i = 1,2$. By the triangle inequality, for any Banach space valued differentiable function w we have

$$\frac{d}{dt}\|w(t)\| \le \left\|\frac{d}{dt}w(t)\right\| \ .$$

Combining (5.1), (5.2) and (5.3),

$$\begin{aligned}
\frac{d}{dt}(\|u_1(t)\|_{\rho_1}^2 &+ \|u_2(t)\|_{\rho_2}^2) \\
&= 2(\|u_1(t)\|_{\rho_1}\frac{d}{dt}\|u_1(t)\|_{\rho_1} + \|u_2(t)\|_{\rho_2}\frac{d}{dt}\|u_2(t)\|_{\rho_2}) \\
&\le 2(\|u_{1t}(t)\|_{\rho_1}\|u_1(t)\|_{\rho_1} + \|u_{2t}(t)\|_{\rho_2}\|u_2(t)\|_{\rho_2}) \\
&\le \|u_{1t}(t)\|_{\rho_1}^2 + \|u_1(t)\|_{\rho_1}^2 + \|u_{2t}(t)\|_{\rho_2}^2 + \|u_2(t)\|_{\rho_2}^2 \\
&\le C(\|P_1 u_{1t}(t)\|^2 + \|P_2 u_{2t}(t)\|^2) + \|u_1(t)\|_{\rho_1}^2 + \|u_2(t)\|_{\rho_2}^2 \\
&\le C[E(0) + (2k-1)(\|u_1(t)\|^2 + \|u_2(t)\|^2)] + \|u_1(t)\|_{\rho_1}^2 + \|u_2(t)\|_{\rho_2}^2 \\
&\le CE(0) + (C(2k-1)+1)(\|u_1(t)\|_{\rho_1}^2 + \|u_2(t)\|_{\rho_2}^2)
\end{aligned}$$

where $C = \max(C_1, C_2)$. Gronwall's lemma implies that $\|u_1(t)\|_{\rho_1} + \|u_2(t)\|_{\rho_2}$ stays bounded in $[0,T)$. Since $\rho_i = \frac{r_i}{2} - 1 > \frac{1}{2}$, $\|u_1(t)\|_\infty + \|u_2(t)\|_\infty$ also stays bounded in $[0,T)$. By Lemma 2.6, a global solution exists. $\square$

5.2. Kernels with Singularity

In the next theorem we will consider kernels of the form $\beta_1(x) = \beta_2(x) = \gamma(|x|)$ where $\gamma \in C^2([0,\infty))$, $\gamma(0) > 0$, $\gamma'(0) < 0$ and $\gamma'' \in L^1 \cap L^\infty$. Then the β_i will have a jump in the first derivative. The typical example we have in mind is the Green's function $\frac{1}{2}e^{-|x|}$. For such kernels we have

$$\widehat{\beta}_i(\xi) \le C_i\left(1+\xi^2\right)^{-1}$$

so $r_1 = r_2 = 2$. Due to the jump in β_i' at $x = 0$, the distributional derivative will satisfy

$$\beta_i'' = \gamma'' + 2\gamma'(0)\delta, \qquad i = 1,2$$

where δ is the Dirac measure and we use the notation $\gamma''(x) = \gamma''(|x|)$. Then we have

$$(\beta_i * w)_{xx} = \gamma'' * w - \lambda w, \qquad i = 1,2$$

where $\lambda = -2\gamma'(0) > 0$. We will call these type of kernels *mildly singular*. For such kernels we extend the global existence result in [14] to the coupled system.

Theorem 5.2. *Let $s > 1/2$ and let the kernels $\beta_1 = \beta_2$ be mildly singular as defined above. Suppose that $\varphi_1, \varphi_2, \psi_1, \psi_2 \in H^s$, $P_1\psi_1, P_2\psi_2 \in L^2$ and $G(\varphi_1, \varphi_2) \in L^1$. If there is some $C > 0$, $k \geq 0$ and $q_i > 1$ so that*

$$|g_i(a,b)|^{q_i} \leq C[G(a,b) + k(a^2 + b^2)]$$

for all $a, b \in \mathbb{R}$ and $i = 1,2$; then the Cauchy problem (1.1)-(1.4) has a global solution with u_1 and u_2 in $C^2([0,\infty), H^s)$.

Proof. By Theorem 2.4 we have a local solution. Suppose the solution (u_1, u_2) exists for $t \in [0,T)$. For fixed $x \in \mathbb{R}$ we define

$$e(t) = \frac{1}{2}[(u_{1t}(x,t))^2 + (u_{2t}(x,t))^2] + \frac{\lambda}{2}[(u_1(x,t))^2 + (u_2(x,t))^2 + 2G(u_1(x,t), u_2(x,t))].$$

Then

$$\begin{aligned}
e'(t) &= [u_{1tt} + \lambda(u_1 + g_1(u_1,u_2))]u_{1t} + [u_{2tt} + \lambda(u_2 + g_2(u_1,u_2))]u_{2t} \\
&= [(\beta_1 * (u_1 + g_1(u_1,u_2)))_{xx} + \lambda(u_1 + g_1(u_1,u_2))]u_{1t} \\
&\quad + [(\beta_2 * (u_2 + g_2(u_1,u_2)))_{xx} + \lambda(u_2 + g_2(u_1,u_2))]u_{2t} \\
&= (\gamma'' * u_1)u_{1t} + (\gamma'' * g_1(u_1,u_2))u_{1t} + (\gamma'' * u_2)u_{2t} + (\gamma'' * g_2(u_1,u_2))u_{2t} \\
&\leq (u_{1t})^2 + (u_{2t})^2 + \frac{1}{2}(\|\gamma'' * u_1\|_\infty^2 + \|\gamma'' * u_2\|_\infty^2) \\
&\quad + \frac{1}{2}(\|\gamma'' * g_1(u_1,u_2)\|_\infty^2 + \|\gamma'' * g_2(u_1,u_2)\|_\infty^2).
\end{aligned}$$

Since $\gamma'' \in L^1 \cap L^\infty$ we have $\gamma'' \in L^p$ for all $p \geq 1$. By Young's inequality

$$\begin{aligned}
e'(t) &\leq (u_{1t})^2 + (u_{2t})^2 + \frac{1}{2}\|\gamma''\|^2 (\|u_1\|^2 + \|u_2\|^2) \\
&\quad + \frac{1}{2}\|\gamma''\|_{L^{p_1}}^2 \|g_1(u_1,u_2)\|_{L^{q_1}}^2 + \frac{1}{2}\|\gamma''\|_{L^{p_2}}^2 \|g_2(u_1,u_2)\|_{L^{q_2}}^2,
\end{aligned}$$

where $1/p_i + 1/q_i = 1$ $(i = 1,2)$. Now the terms may be estimated as

$$\|u_1\|^2 + \|u_2\|^2 \leq E(0)$$

and for $i = 1,2$

$$\begin{aligned}
\|g_i(u_1,u_2)\|_{L^{q_i}}^2 &= \left(\int |g_i(u_1,u_2)|^{q_i}\, dx\right)^{2/q_i} \\
&\leq \left(C\int [G(u_1,u_2) + k(a^2+b^2)]dx\right)^{2/q_i} \leq [C(1+k)E(0)]^{2/q_i}
\end{aligned}$$

so that

$$e'(t) \leq D + 2e(t)$$

for some constant D depending on $\|\gamma''\|_{L^{p_i}}$, $\|\gamma''\|$ and $E(0)$ $(i = 1, 2)$. This inequality holds for all $x \in \mathbb{R}$, $t \in [0, T)$. Gronwall's lemma then implies that $e(t)$ and thus $u_1(x, t)$ and $u_2(x, t)$ stay bounded. Thus by Lemma 2.6 we have global solution. □

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