The Cauchy problem for a class of two-dimensional nonlocal nonlinear wave equations governing anti-plane shear motions in elastic materials

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Abstract. This paper is concerned with the analysis of the Cauchy problem of a general class of two-dimensional nonlinear nonlocal wave equations governing anti-plane shear motions in nonlocal elasticity. The nonlocal nature of the problem is reflected by a convolution integral in the space variables. The Fourier transform of the convolution kernel is nonnegative and satisfies a certain growth condition at infinity. For initial data in L^2 Sobolev spaces, conditions for global existence or finite time blow-up of the solutions of the Cauchy problem are established.

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1. Introduction

In the present paper we consider the initial value problem

$$w_{tt} = \left(\beta * \frac{\partial F}{\partial w_x}\right)_x + \left(\beta * \frac{\partial F}{\partial w_y}\right)_y, \quad (x,y) \in \mathbb{R}^2, \quad t > 0, \tag{1.1}$$

$$w(x,y,0) = \varphi(x,y), \quad w_t(x,y,0) = \psi(x,y), \tag{1.2}$$

where (1.1) models anti-plane shear motions in nonlinear nonlocal elasticity, in terms of non-dimensional quantities. In (1.1)-(1.2), $w = w(x,y,t)$ represents the out-of-plane displacement, the strain energy density function F is a nonlinear function of $|\nabla w|^2 \equiv (w_x^2 + w_y^2)$ for isotropic materials with $F(0) = 0$, and the subscripts denote partial derivatives. The terms with β in (1.1) incorporate the nonlocal effects where

$$(\beta * u)(x,y) = \int_{\mathbb{R}^2} \beta(x - x', y - y')u(x', y')dx'dy'$$

denotes convolution of β and u. The kernel $\beta(x,y)$ is assumed to be an integrable function whose Fourier transform, $\widehat{\beta}(\xi_1,\xi_2)$, satisfies

$$0 \leq \widehat{\beta}(\xi) \leq C(1 + |\xi|^2)^{-r/2}, \quad \text{for all } \xi = (\xi_1,\xi_2), \tag{1.3}$$

where C is a positive constant and $r \geq 2$. The aim of this paper is to establish the well-posedness of the initial value problem (1.1)-(1.2), as well as the global existence and blow-up of solutions for a wide class of the kernel functions $\beta(x,y)$. The number r in (1.3) is closely related to the smoothness of β and, consequently, as the decay rate r gets larger the regularizing effect of the nonlocal behavior increases. This situation is clearly observed through a comparison of Theorem 3.7 and Theorem 3.8.

Although the model requires the nonlinearity to be of the isotropic form $F(w_x^2 + w_y^2)$, our results also apply to the more general $F(w_x, w_y)$-type nonlinearities corresponding to the anisotropic case. The bulk of our work deals with the isotropic form of F but in a separate section we present all the necessary modifications corresponding to the anisotropic form. Similarly, in the model the kernel β is a function of the modulus $|(x,y)|$, but we do not require this restriction on β in our work. Basically, the approach presented here extends the techniques used for the one-dimensional nonlinear nonlocal Boussinesq-type wave equations in the previous studies [1, 2, 3] to the two-dimensional wave equation given by (1.1). It is worthwhile observing that when β is taken as the Dirac measure in (1.1), one recovers the quasilinear wave equation for anti-plane shear motions of the local (classical) theory of elasticity. A natural question is what happens if (1.1)-(1.2) is considered on a bounded domain. This question requires a careful interpretation of the convolution integral and of possible boundary conditions. We refer to the recent manuscript [4] where such interpretations are studied for nonlocal diffusion problems on bounded domains.

The plan of the paper is as follows: In Section 2 we give a brief formulation of the anti-plane shearing problem of nonlocal elasticity. In Section 3 we present a local existence theory for solutions of the Cauchy problem (1.1)-(1.2) for given initial data in suitable Sobolev spaces. In Section 4 we prove global existence of solutions of (1.1)-(1.2) assuming some positivity condition on the nonlinear function F together with enough smoothness on the initial data. In Section 5 we discuss finite time blow-up of solutions. Finally, in Section 6 we show that, with certain modifications, all the results of Sections 2-5 are also valid for more general $F(w_x, w_y)$-type nonlinearities.

Throughout the paper, $\widehat{u}(\xi) = \mathcal{F}(u)(\xi) = \int_{\mathbb{R}^2} e^{-iz\cdot\xi}u(z)dz$ and $\mathcal{F}^{-1}(\widehat{u})(z) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{iz\cdot\xi}\widehat{u}(\xi)d\xi$ denote the Fourier transform and inverse Fourier transform, respectively, where $z = (x, y)$, $\xi = (\xi_1, \xi_2)$, $dz = dxdy$, and $d\xi = d\xi_1 d\xi_2$. Furthermore, $H^s(\mathbb{R}^2)$ denotes the L^2 Sobolev space on $\mathbb{R}^2$. For the H^s norm we use the Fourier transform representation $\|u\|_s^2 = \int_{\mathbb{R}^2}(1 + |\xi|^2)^s|\widehat{u}(\xi)|^2 \mathrm{d}\xi$. Also, $\|u\|_{L^\infty}$ and $\|u\|$ indicate the L^∞ and L^2 norms, respectively, and $\langle u, v\rangle$ refers to the inner product of u and v in $L^2(\mathbb{R}^2)$.

2. The Model

In this section we discuss how equation (1.1) can be derived to describe the propagation of a finite amplitude transverse wave in a nonlocally elastic medium. Before stating our derivation, we need to introduce the concept of nonlocal elasticity. One of the major drawbacks in the local theory of elasticity is that it does not include any intrinsic length scale and consequently does not take into account the long range forces that become increasingly important at small scales. As a result, the local theory of elasticity is incapable of predicting, for instance, (i) the dispersive nature of harmonic waves in crystal lattices and (ii) the boundedness of the stress field near the tip of a crack. In order to overcome such deficiencies various generalizations of the local theory of elasticity have been proposed. One such generalization is the theory of nonlocal elasticity which has been developed by Kröner [5], Eringen and Edelen [6], Kunin [7], Rogula [8], Eringen [9, 10] over the last several decades (For more recent studies on the subject of generalized theories of elasticity, see, for instance, [11, 12, 13, 14, 15, 16] and references therein). What distinguishes the theory of nonlocal elasticity from the local theory of elasticity is that the stress at a point depends on the strain field at every point in the body. Although there has been a considerable amount of research done on small scale effects within the context of the theory of nonlocal elasticity, they are mostly restricted to linear models. Recently, in [1, 2, 3] various Cauchy problems based on a one-dimensional nonlinear model of nonlocal elasticity have been studied. Here, we show how the approach in those studies is extended to the dynamic anti-plane shearing problem of nonlinear nonlocal elasticity.

Consider an isotropic homogeneous nonlocally elastic medium. Identify a material point $\mathbf{X}$ of the medium by its rectangular Cartesian coordinates in a reference configuration: $\mathbf{X} = (X_1, X_2, X_3)$. We assume that the reference configuration is unstressed. Let $\mathbf{x}(\mathbf{X}, t) = (x_1(\mathbf{X}, t), x_2(\mathbf{X}, t), x_3(\mathbf{X}, t))$ denote the position of the same point at time t. Then the displacement and the deformation gradient are given by $\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$ and $\mathbf{A}(\mathbf{X}, t) = \mathrm{Grad}\ \mathbf{x}(\mathbf{X}, t)$, respectively. We suppose that a (local) strain energy density function $F(\mathbf{A})$ per unit volume of the undeformed reference configuration exists, i.e., the material is (locally) hyperelastic, and that it sustains a nontrivial anti-plane shear motion. In the local theory of elasticity, a constitutive equation of the form $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{A}) \equiv \partial F(\mathbf{A})/\partial \mathbf{A}$ holds for a hyperelastic material (see equation (4.3.7) of [17]), where $\boldsymbol{\sigma}$ is the nominal stress tensor (note that some authors use its transpose referred to as the first Piola-Kirchhoff stress tensor). In the theory of nonlocal elasticity the (nonlocal) stress tensor $\mathbf{S}$ is related to the (local) stress tensor $\boldsymbol{\sigma}$ through the constitutive relation $\mathbf{S} = \mathbf{S}(\mathbf{X}, t) \equiv \int \beta(|\mathbf{X} - \mathbf{Y}|)\boldsymbol{\sigma}(\mathbf{A}(\mathbf{Y}, t))d\mathbf{Y}$ where $\beta(|\mathbf{X} - \mathbf{Y}|)$ is a kernel function that weights the contribution of the local stresses to the nonlocal stresses. In the absence of body forces, the (Lagrangian) equation of motion (see equation (3.4.4) of [17]) is given by $\rho_0 \ddot{\mathbf{x}} = \mathrm{Div}\, \mathbf{S}$ where ρ_0 is the mass density of the medium and a superposed dot indicates the material time derivative. The only difference between the equations of the local theory of elasticity and those of the nonlocal model presented here is due to the constitutive equations.

Now we consider an anti-plane shear motion of the form

$$x_1 = X_1, \qquad x_2 = X_2, \qquad x_3 = X_3 + w(X_1, X_2, t) \tag{2.1}$$

for a nonlocally elastic material, where the out-of-plane displacement w is the only non-zero component of displacement, i.e. $u_1 = u_2 \equiv 0$ and $u_3 \equiv w(X_1, X_2, t)$. We henceforth replace the arguments X_1 and X_2 of the displacement w with x and y, respectively, and denote partial differentiations with subscript letters. For isotropic materials the strain energy density function F is a function of the three fundamental scalar invariants of the left Cauchy-Green matrix and for the anti-plane shear motion (2.1) it turns out to be a function of $w_x^2 + w_y^2$ alone: $F = F(w_x^2 + w_y^2)$ (see, for instance, Section 4 of [18]). Furthermore, the equation of motion reduces to the scalar partial differential equation $\rho_0 w_{tt} = (\beta * \sigma_{13})_x + (\beta * \sigma_{23})_y$ where σ_{13} and σ_{23} are the (local) shear stresses arising due to the anti-plane shear motion and they are given by $\sigma_{13} = \partial F/\partial w_x$ and $\sigma_{23} = \partial F/\partial w_y$. The computations are identical to those in the conventional formulation of nonlinear elasticity, provided we replace the nonlocal stress tensor with the local stress of conventional theory of elasticity [18]. The nonlocal behavior is represented by the convolution integral. Thus, without loss of generality, if we make a suitable non-dimensionalization of the equation of motion (see [2] for the non-dimensionalization in the one-dimensional case) and use the same symbols to avoid a proliferation of notation, or simply take the mass density to be 1, we get (1.1). Equation (1.1) is consistent with that of the conventional formulation of nonlinear elasticity. In other words, when β is taken as the Dirac measure to eliminate the nonlocal effect, (1.1) reduces to the quasilinear wave equation governing anti-plane shear motions in the local theory of nonlinear elasticity (see for instance equation (7.10) of [18] or equation (2.2) of [19]). A list of the most commonly used one-dimensional kernel functions that satisfy the one-dimensional version of the condition given in (1.3) is presented in [2]. We now present three examples of two-dimensional kernel functions used in the literature.

(i) *The Gaussian kernel* [20]: $\beta(x, y) = (2\pi)^{-1} e^{-(x^2+y^2)/2}$. We have $\widehat{\beta}(\xi_1, \xi_2) = e^{-(\xi_1^2+\xi_2^2)/2}$. This is a highly regularizing kernel as can be observed by the fact that we can take any r in (1.3).

(ii) The modified Bessel function kernel [20]: $\beta(x,y) = (2\pi)^{-1}K_0(\sqrt{x^2+y^2})$ where K_0 is the modified Bessel function of the second kind of order zero. Since $\widehat{\beta}(\xi_1,\xi_2) = (1+\xi_1^2+\xi_2^2)^{-1}$, for this special case we have $r = 2$ in (1.3). Note that β is the Green's function for the operator $(1-\Delta)$ where Δ denotes the two-dimensional Laplacian. In this case (1.1) can equivalently be written as

$$w_{tt} - \Delta w_{tt} = \left(\frac{\partial F}{\partial w_x}\right)_x + \left(\frac{\partial F}{\partial w_y}\right)_y .$$

Letting $F(s) = \frac{1}{2}s + G(s)$ we obtain the more familiar form

$$w_{tt} - \Delta w - \Delta w_{tt} = \left(\frac{\partial G}{\partial w_x}\right)_x + \left(\frac{\partial G}{\partial w_y}\right)_y .$$

(iii) The bi-Helmholtz type kernel [21]:

$$\beta(x,y) = \frac{1}{2\pi(c_1^2-c_2^2)}[K_0(\sqrt{x^2+y^2}/c_1) - K_0(\sqrt{x^2+y^2}/c_2)]$$

where c_1 and c_2 are real and positive constants. Since $\widehat{\beta}(\xi_1,\xi_2) = [1+\gamma_1(\xi_1^2+\xi_2^2)+\gamma_2(\xi_1^2+\xi_2^2)^2]^{-1}$ where $\gamma_1 = c_1^2+c_2^2$ and $\gamma_2 = c_1^2c_2^2$ we have $r = 4$. As above, β is Green's function for the operator $(1-\gamma_1\Delta+\gamma_2\Delta^2)$. Then (1.1) becomes

$$w_{tt} - \Delta w - \gamma_1\Delta w_{tt} + \gamma_2\Delta^2 w_{tt} = \left(\frac{\partial G}{\partial w_x}\right)_x + \left(\frac{\partial G}{\partial w_y}\right)_y .$$

In the remainder of this paper we discuss the question of well-posedness of the Cauchy problem (1.1)-(1.2).

3. Local Existence and Uniqueness of Solutions

In the present section, we prove existence and uniqueness of solutions over a small time interval. Local well-posedness is established by converting the initial value problem (1.1)-(1.2) into a system of Banach space-valued ordinary differential equations. Thus (1.1)-(1.2) is formally equivalent to the system

$$w_t = v, \quad w(0) = \varphi, \tag{3.1}$$

$$v_t = Kw, \quad v(0) = \psi \tag{3.2}$$

where the operator K is defined as

$$Kw = \left(\beta * \frac{\partial F}{\partial w_x}\right)_x + \left(\beta * \frac{\partial F}{\partial w_y}\right)_y . \tag{3.3}$$

The Banach space X^s will be defined as

$$X^s = \{w \in H^s(\mathbb{R}^2);\ w_x, w_y \in L^\infty(\mathbb{R}^2)\},$$

endowed with the norm

$$\|w\|_{X^s} = \|w\|_s + \|w_x\|_{L^\infty} + \|w_y\|_{L^\infty}. \tag{3.4}$$

The following two lemmas are useful in the proof.

Lemma 3.1 *(Sobolev Embedding Theorem) If* $s > \frac{n}{2} + k$, *then* $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

In particular when $n = 2$ and $s > 2$, the embedding in Lemma 3.1 implies the norm estimate $\| \, |\nabla u| \, \|_{L^\infty} \leq C\|u\|_s$. We refer to Chapter 5 of [22] for a discussion on the many versions of the Sobolev embedding theorem.

Lemma 3.2 *Let* $s \geq 0$ *and let* $u_1, u_2 \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Then* $u_1 u_2 \in H^s(\mathbb{R}^n)$ *and*

$$\|u_1 u_2\|_s \leq C(\|u_1\|_s \|u_2\|_{L^\infty} + \|u_1\|_{L^\infty} \|u_2\|_s).$$

Lemma 3.2 can be found in [23] in a more general L^p-setting (see Lemma X4 of [23]); we also refer to [24] for a general discussion. The following two lemmas (see Chapter 5 of [25]) have been used by many authors (for instance, see Lemmas 1, 2 and 3 of [26] or Lemmas 2.3 and 2.4 of [27]) to control the nonlinear terms.

Lemma 3.3 *Let* $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R})$ *with* $f(0) = 0$. *Then for any* $u \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, *we have* $f(u) \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Moreover there is some constant* $A(M)$ *depending on* M *(and* s*) such that for all* $u \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *with* $\|u\|_{L^\infty} \leq M$

$$\|f(u)\|_s \leq A(M)\|u\|_s \ .$$

Lemma 3.4 *Let* $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R})$. *Then for any* $M > 0$ *there is some constant* $B(M)$ *depending on* M *(and* s*) such that for all* $u_1, u_2 \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *with* $\|u_1\|_{L^\infty} \leq M$, $\|u_2\|_{L^\infty} \leq M$ *and* $\|u_1\|_s \leq M$, $\|u_2\|_s \leq M$ *we have*

$$\|f(u_1) - f(u_2)\|_s \leq B(M)\|u_1 - u_2\|_s \ .$$

In our case the nonlinearities are of the form

$$\frac{\partial F}{\partial w_x}(|\nabla w|^2) = 2w_x F'(|\nabla w|^2), \qquad \frac{\partial F}{\partial w_y}(|\nabla w|^2) = 2w_y F'(|\nabla w|^2)$$

where F' denotes the derivative of F. It follows from repeated applications of Lemma 3.2 that for the above terms Lemmas 3.3 and 3.4 take the following forms:

Lemma 3.5 *Let* $s \geq 1$, $F \in C^{[s]+1}(\mathbb{R})$. *Then for any* $w \in X^s$, *we have*

$$\frac{\partial F}{\partial w_x}(|\nabla w|^2) \in H^{s-1}(\mathbb{R}^2), \qquad \frac{\partial F}{\partial w_y}(|\nabla w|^2) \in H^{s-1}(\mathbb{R}^2).$$

Moreover there is some constant $A(M)$ *depending on* M *(and* s*) such that for all* $w \in X^s$ *with* $\| \, |\nabla w| \, \|_{L^\infty} \leq M$

$$\|\frac{\partial F}{\partial w_x}(|\nabla w|^2)\|_{s-1} \leq A(M)\|w\|_s,$$
$$\|\frac{\partial F}{\partial w_y}(|\nabla w|^2)\|_{s-1} \leq A(M)\|w\|_s \ .$$

Lemma 3.6 *Let $s \geq 1$, $F \in C^{[s]+1}(\mathbb{R})$. Then for any $M > 0$ there is some constant $B(M)$ depending on M (and s) such that for all $w_1, w_2 \in X^s$ with $\|w_1\|_{X^s} \leq M$, $\|w_2\|_{X^s} \leq M$ we have*

$$\|\frac{\partial F}{\partial w_x}(|\nabla w_1|^2) - \frac{\partial F}{\partial w_x}(|\nabla w_2|^2)\|_{s-1} \leq B(M)\|w_1 - w_2\|_s,$$
$$\|\frac{\partial F}{\partial w_y}(|\nabla w_1|^2) - \frac{\partial F}{\partial w_y}(|\nabla w_2|^2)\|_{s-1} \leq B(M)\|w_1 - w_2\|_s .$$

We want to emphasize that, while the constants $B(M)$ in Lemmas 3.4 and 3.6 are indeed local Lipschitz constants depending on the X^s-norms, the constants $A(M)$ in Lemmas 3.3 and 3.5 depend only on the L^∞-norms. This property of $A(M)$ will be used in the proof of Lemma 3.9 which characterizes the type of blow-up.

When $s > 2$ we have the following local well posedness result.

Theorem 3.7 *Suppose $s > 2$ and the decay rate r in (1.3) satisfies $r \geq 2$. For $\varphi, \psi \in H^s(\mathbb{R}^2)$, there is some $T > 0$ such that the Cauchy problem (1.1)-(1.2) is well posed with solution $w = w(x, y, t)$ in $C^2([0, T], H^s(\mathbb{R}^2))$.*

Proof. Let $w \in H^s(\mathbb{R}^2)$. For $s > 2$, by the Sobolev Embedding Theorem we have $|\nabla w| \in L^\infty(\mathbb{R}^2)$. Thus $X^s = H^s(\mathbb{R}^2)$ and the norm $\|w\|_{X^s}$ can be replaced by the equivalent H^s norm $\|w\|_s$. Since (1.1)-(1.2) is formally equivalent to (3.1)-(3.2), we will use the well-known existence-uniqueness (Picard-Lindelöf) theorem for Banach space valued systems of ordinary differential equations (for instance, see Theorem 5.1.1 of [28]). Obviously, all we need is to show that the operator K of (3.3) is locally Lipschitz on X^s. We first show that K maps X^s into X^s. We estimate the convolution as

$$\|\beta * u\|_s = \|(1 + |\xi|^2)^{s/2}\widehat{\beta}(\xi)\widehat{u}(\xi)\| \leq C\|(1 + |\xi|^2)^{(s-r)/2}\widehat{u}(\xi)\| = C\|u\|_{s-r},$$

where we have used inequality (1.3). By Lemma 3.5 for $\| \, |\nabla w| \, \|_{L^\infty} \leq M$

$$\begin{aligned} \| \left(\beta * \frac{\partial F}{\partial w_x}\right)_x \|_s &\leq \|\beta * \frac{\partial F}{\partial w_x}\|_{s+1} \leq C\|\frac{\partial F}{\partial w_x}\|_{s+1-r} \\ &\leq CA(M)\|w\|_{s+2-r} \leq CA(M)\|w\|_s \end{aligned}$$

where we have used $r \geq 2$. The same holds for the term $\left(\beta * \frac{\partial F}{\partial w_y}\right)_y$ and

$$\|Kw\|_s \leq CA(M)\|w\|_{s+2-r} \leq CA(M)\|w\|_s. \tag{3.5}$$

Similarly, for $w_1, w_2 \in X^s$ with $\|w_1\|_s \leq M$ and $\|w_2\|_s \leq M$, by Lemma 3.6

$$\begin{aligned} \| \left(\beta * \frac{\partial F}{\partial w_x}(|\nabla w_1|^2)\right)_x - \left(\beta * \frac{\partial F}{\partial w_x}(|\nabla w_2|^2)\right)_x \|_s &\leq CB(M)\|w_1 - w_2\|_{s+2-r} \\ &\leq CB(M)\|w_1 - w_2\|_s. \end{aligned}$$

As above, the same holds for the term $\left(\beta * \frac{\partial F}{\partial w_y}\right)_y$. So, K is locally Lipschitz on X^s and thus the local well posedness of the Cauchy problem is established. ∎

When $r > 3$ in (1.3), the extra regularizing effect of β allows us to improve the result in Theorem 3.7 to the case of $s \geq 1$.

Theorem 3.8 *Suppose* $s \geq 1$ *and the decay rate* r *in (1.3) satisfies* $r > 3$. *For* $\varphi, \psi \in X^s$, *there is some* $T > 0$ *such that the Cauchy problem (1.1)-(1.2) is well posed with solution* $w = w(x, y, t)$ *in* $C^2([0, T], X^s)$.

Proof. Similar to the proof of Theorem 3.7 it suffices to show that the map K given in (3.3) is locally Lipschitz on X^s. Recall that $\|Kw\|_{X^s} = \|Kw\|_s + \|(Kw)_x\|_{L^\infty} + \|(Kw)_y\|_{L^\infty}$. The term $\|Kw\|_s$ can be estimated by $\|w\|_s$ as above. For $\epsilon = r - 3 > 0$ we have

$$\begin{aligned} \|(Kw)_x\|_{L^\infty} &\leq C\|(Kw)_x\|_{1+\epsilon} \leq C\|Kw\|_{2+\epsilon} \leq C\|Kw\|_{s+1+\epsilon} \\ &\leq CA(M)\|w\|_{s+3+\epsilon-r} = CA(M)\|w\|_s. \end{aligned} \tag{3.6}$$

where we have used (3.5) and the Sobolev Embedding Theorem. The same holds for $(Kw)_y$ and a similar estimate as in the proof of Theorem 3.7 shows that K is locally Lipschitz on X^s. ■

The solution of (1.1)-(1.2) can be extended to a maximal interval $[0, T_{\max})$ where finite $T_{\max}$ is characterized by the blow up condition

$$\limsup_{t \to T_{\max}^-} (\|w(t)\|_{X^s} + \|w_t(t)\|_{X^s}) = \infty.$$

Obviously $T_{\max} = \infty$, i.e. there is a global solution, if and only if for any $T < \infty$

$$\limsup_{t \to T^-} (\|w(t)\|_{X^s} + \|w_t(t)\|_{X^s}) < \infty.$$

The lemma below characterizes the type of blow-up; namely blow-up occurs in the L^∞-norm of $|\nabla w|$.

Lemma 3.9 *Suppose that the conditions of Theorem 3.7 or Theorem 3.8 hold. Then* $T_{\max} = \infty$, *i.e. there is a global solution of the Cauchy problem (1.1)-(1.2), if and only if for any* $T > 0$

$$\limsup_{t \to T^-} (\|w_x(t)\|_{L^\infty} + \|w_y(t)\|_{L^\infty}) < \infty.$$

Proof. Since

$$\|w_x(t)\|_{L^\infty} + \|w_y(t)\|_{L^\infty} \leq \|w(t)\|_{X^s},$$

it suffices to prove that if the solution exists for $t \in [0, T)$ and $\|w_x(t)\|_{L^\infty} + \|w_y(t)\|_{L^\infty} \leq M$ for all $0 \leq t < T$ then both $\|w(t)\|_{X^s}$ and $\|w_t(t)\|_{X^s}$ stay bounded. Integrating equation (1.1) twice and calculating the resulting double integral as an iterated integral, we obtain

$$w(t) = \varphi + t\psi + \int_0^t (t - \tau)(Kw)(\tau) d\tau, \tag{3.7}$$

$$w_t(t) = \psi + \int_0^t (Kw)(\tau) d\tau. \tag{3.8}$$

But, by (3.5), $\|(Kw)(\tau)\|_s \leq CA(M)\|w(\tau)\|_{s+2-r} \leq CA(M)\|w(\tau)\|_s$ where the constant $A(M)$ depends only on M. Hence

$$\|w(t)\|_s + \|w_t(t)\|_s \leq \|\varphi\|_s + (1 + T)\|\psi\|_s + (1 + T)CA(M) \int_0^t \|w(\tau)\|_s d\tau,$$

and Gronwall's Lemma gives

$$\|w(t)\|_s + \|w_t(t)\|_s \leq (\|\varphi\|_s + (1+T)\|\psi\|_s)e^{(1+T)CA(M)T} \tag{3.9}$$

for all $t \in [0,T)$. We now estimate $\|w_{tx}(t)\|_{L^\infty}$. The estimate for $\|w_{ty}(t)\|_{L^\infty}$ follows similarly. In the case of Theorem 3.7 (where $s > 2$), by the Sobolev Embedding Theorem,

$$\|w_{tx}(t)\|_{L^\infty} \leq C\|w_t(t)\|_s$$

so that (3.9) applies. In the case of Theorem 3.8 (where $r > 3$), from (3.8)

$$\|w_{tx}(t)\|_{L^\infty} \leq \|\psi_x\|_{L^\infty} + \int_0^t \|(Kw)_x(\tau)\|_{L^\infty}\, d\tau. \tag{3.10}$$

By (3.6)

$$\|(Kw)_x(\tau)\|_{L^\infty} \leq CA(M)\|w(\tau)\|_s,$$

and by (3.9)

$$\|w(\tau)\|_s \leq (\|\varphi\|_s + (1+T)\|\psi\|_s)e^{(1+T)CA(M)T}.$$

Finally, plugging into (3.10) we obtain the required estimate

$$\|w_{tx}(t)\|_{L^\infty} \leq \|\psi_x\|_{L^\infty} + TCA(M)(\|\varphi\|_s + (1+T)\|\psi\|_s)e^{(1+T)CA(M)T},$$

which corresponds to the case $s \geq 1$, $r > 3$. ■

4. Conservation of Energy and Global Existence

In the present section we will prove that locally well defined solutions can be extended to the entire time.

In the study of global existence of solutions the conservation of energy plays a key role. First, time invariance of the energy functional will be shown. To this end, we define an unbounded linear operator R on $H^s(\mathbb{R}^2)$ as $Ru = \mathcal{F}^{-1}\left((\widehat{\beta}(\xi))^{-\frac{1}{2}}\widehat{u}(\xi)\right)$ where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and $\widehat{\beta}(\xi)$ is defined in (1.3). Obviously, $R^p u = \mathcal{F}^{-1}\left((\widehat{\beta}(\xi))^{-\frac{p}{2}}\widehat{u}(\xi)\right)$ for a real number p with u in the domain of R^p. On the other hand, $R^{-2}u = \beta * u$ and formally (1.1) can be rewritten as

$$R^2 w_{tt} = \left(\frac{\partial F}{\partial w_x}\right)_x + \left(\frac{\partial F}{\partial w_y}\right)_y. \tag{4.1}$$

Here we have used the fact that convolution commutes with differentiation in the distribution sense, i.e. $(\beta * u)_x = \beta * u_x$.

Lemma 4.1 *Suppose that the conditions of Theorem 3.7 or Theorem 3.8 hold and the solution of the Cauchy problem (1.1)-(1.2) exists in $C^2([0,T), X^s)$. If $R\psi \in L^2(\mathbb{R}^2)$, then $Rw_t(t) \in L^2(\mathbb{R}^2)$ for all $t \in [0,T)$. Moreover, if $R\varphi \in L^2(\mathbb{R}^2)$, then $Rw(t) \in L^2(\mathbb{R}^2)$ for all $t \in [0,T)$.*

Proof. Formally, from (3.8) we have

$$Rw_t(t) = R\psi + \int_0^t (RKw)(\tau)d\tau \ . \tag{4.2}$$

Note that

$$RKw = \left(\alpha * \frac{\partial F}{\partial w_x}\right)_x + \left(\alpha * \frac{\partial F}{\partial w_y}\right)_y ,$$

where $\widehat{\alpha}(\xi) = (\widehat{\beta}(\xi))^{1/2}$. Then similar to the derivation of (3.5) (replacing β by α and hence r by $r/2$) we get

$$\|(RKw)(\tau)\|_{s+\frac{r}{2}-2} \leq C\|w(\tau)\|_s.$$

Since either ($s > 2$ and $r \geq 2$) or ($s \geq 1$ and $r > 3$), in both cases we have $s + \frac{r}{2} - 2 > 0$. Thus the right-hand side of (4.2) belongs to $L^2(\mathbb{R}^2)$ and the conclusion follows. The second statement follows similarly from (3.7). ■

Lemma 4.2 *Suppose that the solution of the Cauchy problem (1.1)-(1.2) exists on some interval* $[0,T)$*. If* $R\psi \in L^2(\mathbb{R}^2)$ *and the function* $F(|\nabla\varphi|^2)$ *belongs to* $L^1(\mathbb{R}^2)$*, then for any* $t \in [0,T)$ *the energy*

$$E(t) = \frac{1}{2}\|Rw_t(t)\|^2 + \int_{\mathbb{R}^2} F(|\nabla w(t)|^2)dxdy \tag{4.3}$$

is constant in $[0,T)$*.*

Proof. By Lemma 4.1, $Rw_t(t) \in L^2(\mathbb{R}^2)$. Multiplying (4.1) by w_t, integrating in x and y, and using Parseval's identity we get

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{\mathbb{R}^2} \frac{1}{2}(\widehat{\beta}(\xi))^{-1}|\widehat{w}_t(\xi,t)|^2 d\xi + \int_{\mathbb{R}^2} F'(|\nabla w(t)|^2)\frac{\partial}{\partial t}(|\nabla w(t)|^2)dxdy \\ &= \frac{d}{dt}\int_{\mathbb{R}^2}\left(\frac{1}{2}(Rw_t(t))^2 + F(|\nabla w(t)|^2)\right)dxdy \end{aligned}$$

which implies the conservation of energy. ■

The main result of this section is the following theorem.

Theorem 4.3 *Suppose* $s \geq 1$ *and the decay rate* r *in (1.3) satisfies* $r > 4$*. Let* $\varphi, \psi \in X^s$*,* $R\psi \in L^2(\mathbb{R}^2)$ *and* $F(|\nabla\varphi|^2) \in L^1(\mathbb{R}^2)$*. If there is some* $k > 0$ *so that* $F(u) \geq -ku$ *for all* $u \geq 0$*, then the Cauchy problem (1.1)-(1.2) has a global solution in* $C^2([0,\infty), X^s)$*.*

Proof. By Theorem 3.8 the Cauchy problem is locally well-posed. Assume $w \in C^2([0,T), X^s)$ for some $T > 0$. Since $F(u) \geq -ku$, for all $t \in [0,T)$ we have

$$\begin{aligned} \|Rw_t(t)\|^2 &= 2E(0) - 2\int_{\mathbb{R}^2} F(|\nabla w(t)|^2)dxdy, \\ &\leq 2E(0) + 2k\int_{\mathbb{R}^2} |\nabla w(t)|^2 dxdy, \end{aligned} \tag{4.4}$$

where $E(0)$ is the initial energy. On the other hand we have

$$\begin{aligned}\|Rw_t(t)\|^2 &= \int_{\mathbb{R}^2} (\widehat{\beta}(\xi))^{-1} |\widehat{w}_t(\xi,t)|^2 d\xi, \\ &\geq C^{-1} \int_{\mathbb{R}^2} (1+|\xi|^2)^{\frac{r}{2}} |\widehat{w}_t(\xi,t)|^2 d\xi, \\ &= C^{-1} \|w_t(t)\|^2_{\frac{r}{2}}, \end{aligned} \tag{4.5}$$

where (1.3) is used. Combining (4.4) and (4.5)

$$\begin{aligned}\frac{d}{dt}\|w(t)\|^2_{\frac{r}{2}} &\leq 2\|w(t)\|_{\frac{r}{2}} \|w_t(t)\|_{\frac{r}{2}} \\ &\leq \|w(t)\|^2_{\frac{r}{2}} + \|w_t(t)\|^2_{\frac{r}{2}} \\ &\leq \|w(t)\|^2_{\frac{r}{2}} + C\|Rw_t(t)\|^2 \\ &\leq \|w(t)\|^2_{\frac{r}{2}} + 2C(E(0) + k\||\nabla w(t)|\|^2) \\ &\leq 2CE(0) + (1+2Ck)\|w(t)\|^2_{\frac{r}{2}}, \end{aligned}$$

where $\||\nabla w(t)|\| \leq \|w(t)\|_1 \leq \|w(t)\|_{\frac{r}{2}}$ is used. Gronwall's lemma implies that $\|w\,(t)\|_{\frac{r}{2}}$ stays bounded in $[0,T)$. As $r > 4$ we have $\frac{r}{2} - 1 > 1$ and the Sobolev Embedding Theorem implies

$$\||\nabla w(t)|\|_{L^\infty} \leq \||\nabla w(t)|\|_{\frac{r}{2}-1} \leq \|w(t)\|_{\frac{r}{2}}.$$

We conclude that $\||\nabla w(t)|(t)\|_{L^\infty}$ also stays bounded in $[0,T)$. By Lemma 3.9, this implies a global solution. ■

5. Blow up

In this section a blow-up result for (1.1)-(1.2) will be presented. The following lemma, based on the idea of Levine [29], will be used to prove blow up of solutions in finite time for certain nonlinearities and initial data.

Lemma 5.1 *Suppose that* $\mathcal{H}(t)$, $t \geq 0$, *is a positive, twice differentiable function satisfying* $\mathcal{H}''\mathcal{H} - (1+\nu)(\mathcal{H}')^2 \geq 0$ *where* $\nu > 0$. *If* $\mathcal{H}(0) > 0$ *and* $\mathcal{H}'(0) > 0$, *then* $\mathcal{H}(t) \to \infty$ *as* $t \to t_1$ *for some* $t_1 \leq \mathcal{H}(0)/\nu\mathcal{H}'(0)$.

Theorem 5.2 *Suppose that the solution,* w, *of the Cauchy problem (1.1)-(1.2) exists,* $R\varphi$, $R\psi \in L^2(\mathbb{R}^2)$ *and* $F(|\nabla\varphi|^2) \in L^1(\mathbb{R}^2)$. *If there exists a positive number* ν *such that*

$$uF'(u) \leq (1+2\nu)F(u) \quad \textit{for all } u \geq 0, \tag{5.1}$$

and

$$E(0) = \frac{1}{2}\|R\psi\|^2 + \int_{\mathbb{R}^2} F(|\nabla\varphi|^2)dxdy < 0,$$

then the solution, w, *of the Cauchy problem (1.1)-(1.2) blows up in finite time.*

Proof. We assume that the global solution to (1.1)-(1.2) exists. Then, by Lemma 4.1, $Rw(t),\ Rw_t(t) \in L^2(\mathbb{R}^2)$ for all $t > 0$. Let $\mathcal{H}(t) = \|Rw(t)\|^2 + b(t+t_0)^2$ where b and t_0 are positive constants to be determined later. Then we have

$$\begin{aligned}\mathcal{H}' &= 2\langle Rw_t, Rw\rangle + 2b(t+t_0)\\ \mathcal{H}'' &= 2\|Rw_t\|^2 + 2\langle Rw_{tt}, Rw\rangle + 2b.\end{aligned}$$

Note that $\mathcal{H}'(0) = 2\langle R\varphi, R\psi\rangle + 2bt_0 > 0$ for sufficiently large t_0. Using the inequality $uF'(u) \le (1+2\nu)F(u)$ together with (4.4) and (4.5) we have

$$\begin{aligned}\langle Rw_{tt}, Rw\rangle &= \langle R^2 w_{tt}, w\rangle\\ &= -2\int_{\mathbb{R}^2} |\nabla w|^2 F'(|\nabla w|^2)dxdy\\ &\ge -2(1+2\nu)\int_{\mathbb{R}^2} F(|\nabla w|^2)dxdy\\ &= (1+2\nu)\left(\|Rw_t\|^2 - 2E(0)\right),\end{aligned}$$

so that

$$\mathcal{H}'' \ge 4(1+\nu)\|Rw_t\|^2 - 4(1+2\nu)E(0) + 2b.$$

On the other hand, using $2ab \le a^2 + b^2$ and Cauchy-Schwarz inequalities we have

$$\begin{aligned}(\mathcal{H}')^2 &= 4\left[\langle Rw, Rw_t\rangle + b(t+t_0)\right]^2\\ &\le 4\left(\|Rw\|^2\ \|Rw_t\|^2 + 2b(t+t_0)\|Rw\|\ \|Rw_t\| + b^2(t+t_0)^2\right)\\ &\le 4\left(\|Rw\|^2\ \|Rw_t\|^2 + b\|Rw\|^2 + b\|Rw_t\|^2(t+t_0)^2 + b^2(t+t_0)^2\right).\end{aligned}$$

Thus

$$\begin{aligned}&\mathcal{H}''\mathcal{H} - (1+\nu)(\mathcal{H}')^2\\ &\quad\ge \left(4(1+\nu)\|Rw_t\|^2 - 4(1+2\nu)E(0) + 2b\right)\left(\|Rw\|^2 + b(t+t_0)^2\right)\\ &\quad\quad - 4(1+\nu)\left(\|Rw\|^2\ \|Rw_t\|^2 + b\|Rw\|^2 + b\|Rw_t\|^2(t+t_0)^2 + b^2(t+t_0)^2\right)\\ &\quad = -2(1+2\nu)(b+2E(0))\mathcal{H}.\end{aligned}$$

Now if we choose $b \le -2E(0)$, this gives

$$\mathcal{H}''(t)\mathcal{H}(t) - (1+\nu)\left(\mathcal{H}'(t)\right)^2 \ge 0.$$

According to the Blow-up Lemma 5.1, this implies that $\mathcal{H}(t)$, and thus $\|Rw(t)\|^2$ blows up in finite time. ∎

Consider a typical nonlinearity of the form $F(u) = au^q$ with $q > 0$. When $a > 0$, Theorem 4.3 will apply and a global solution exists (for suitable s, r and initial data). When $a < 0$, the blow-up condition (5.1) of Theorem 5.2 holds if and only if $q > 1$. This observation says that the global existence result of Theorem 4.3 is essentially sharp compared to the blow-up result.

Finally, we conclude with a short discussion on the condition $E(0) < 0$. If $F(u) \ge 0$ for all $u \ge 0$, then by Theorem 4.3, there is a global solution. If $F(u)$ is negative on some interval I, we can choose φ with support in I so that $\int_{\mathbb{R}^2} F(|\nabla\varphi|^2)dxdy < 0$. Hence, when $\psi = 0$ or $R\psi$ is sufficiently small we get $E(0) < 0$. This also shows that blow up may occur even for small initial data.

6. The Anisotropic Case

In this section we will consider the more general nonlinear term of the form $F(w_x, w_y)$ with $F(0,0) = 0$ rather than the isotropic form $F(|\nabla w|^2)$. Such a form appears as the strain energy function of anisotropic materials [18]. With minor modifications on the assumptions, all the results of Sections 2-5 can be generalized. Following the layout of the manuscript we will pinpoint out these modifications and briefly explain how the proofs will change accordingly if we replace $F(|\nabla w|^2)$ by $\tilde{F}(w_x, w_y) = \tilde{F}(\nabla w)$ where the symbol $\tilde{}$ is employed to distinguish the anisotropic form of the strain energy function.

Local Existence and Uniqueness of Solutions

The main step in Section 2 is to show that the map K of (3.3) is locally Lipschitz on the Banach space X^s. To deal with the more general nonlinearity $\tilde{F}(\nabla w)$ we will use the vector versions of Lemmas 3.3 and 3.4 [25]. For a vector function $U = (u_1, u_2)$ the notation $\|U\| = \|u_1\| + \|u_2\|$ will be employed where $\|.\|$ denotes a given norm.

Lemma 6.1 *Let $s \geq 0$, $h \in C^{[s]+1}(\mathbb{R}^2)$ with $h(0) = 0$. Then for any $U = (u_1, u_2) \in (H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))^2$, we have $h(U) \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Moreover there is some constant $A(M)$ depending on M such that for all $U \in (H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))^2$ with $\|U\|_{L^\infty} \leq M$*

$$\|h(U)\|_s \leq A(M)\|U\|_s .$$

Lemma 6.2 *Let $s \geq 0$, $h \in C^{[s]+1}(\mathbb{R}^2)$. Then for any $M > 0$ there is some constant $B(M)$ such that for all $U_1, U_2 \in (H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))^2$ with $\|U_1\|_{L^\infty} \leq M$, $\|U_2\|_{L^\infty} \leq M$ and $\|U_1\|_s \leq M$, $\|U_2\|_s \leq M$ we have*

$$\|h(U_1) - h(U_2)\|_s \leq B(M)\|U_1 - U_2\|_s .$$

Through Lemmas 6.1 and 6.2 we obtain the estimates of Lemmas 3.5 and 3.6 for $\tilde{F}(\nabla w)$. With these new estimates, the proofs of Theorems 3.7 and 3.8 on local well posedness and of Lemma 3.9 about the blow up criterion follow exactly the same way for the anisotropic form.

Conservation of Energy and Global Existence

Due to the new estimates on the map K of (3.3), Lemma 4.1 holds also for the general case. In Lemma 4.2 the energy must be replaced by

$$E(t) = \frac{1}{2}\|Rw_t(t)\|^2 + \int_{\mathbb{R}^2} \tilde{F}(\nabla w(t))dxdy.$$

The proof of the energy identity is the same with the obvious modification

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} \tilde{F}(\nabla w(t))dxdy &= \int_{\mathbb{R}^2} \left(\frac{\partial \tilde{F}}{\partial w_x} w_{xt} + \frac{\partial \tilde{F}}{\partial w_y} w_{yt}\right) dxdy \\ &= -\int_{\mathbb{R}^2} \left[\left(\frac{\partial \tilde{F}}{\partial w_x}\right)_x + \left(\frac{\partial \tilde{F}}{\partial w_y}\right)_y\right] w_t dxdy. \end{aligned}$$

In Theorem 4.3, we change the assumption on the nonlinearity as $\tilde{F}(U) \geq -k|U|^2$ for all $U \in \mathbb{R}^2$. Then the estimate (4.4) still holds for the anisotropic case and thus the rest of the proof follows. We note that our new assumption $\tilde{F}(U) \geq -k|U|^2$ reduces to the previous condition $F(u) \geq -ku$ of Theorem 4.3 when the nonlinearity is of the form $F = F(|\nabla w|^2)$.

Blow up

The blow-up condition

$$uF'(u) \leq (1+2\nu)F(u) \quad \text{for all } u \geq 0,$$

of Theorem 5.2 takes the form

$$U \cdot \nabla \tilde{F}(U) \leq 2(1+2\nu)\tilde{F}(U) \quad \text{for all } U \in \mathbb{R}^2$$

for the general nonlinearity $\tilde{F}(\nabla w)$. With this new blow-up condition all the steps in the proof of Theorem 5.2 are still valid for the general case except for the modification below in the estimate for the term $\langle Rw_{tt}, Rw \rangle$:

$$\begin{aligned}
\langle Rw_{tt}, Rw \rangle &= \langle R^2 w_{tt}, w \rangle \\
&= -\int_{\mathbb{R}^2} \nabla w \cdot \nabla \tilde{F}(\nabla w) dxdy \\
&\geq -2(1+2\nu) \int_{\mathbb{R}^2} \tilde{F}(\nabla w) dxdy \\
&= (1+2\nu)\left(\|Rw_t\|^2 - 2E(0)\right).
\end{aligned}$$

We note that the new blow-up condition reduces to the blow-up condition of Theorem 5.2 for the isotropic form $F(|\nabla w|^2)$.

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