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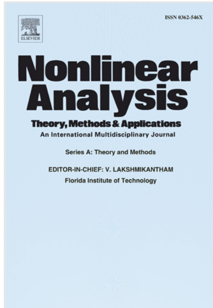

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CLOSING THE GAP IN THE PURELY ELLIPTIC GENERALIZED DAVEY-STEWARTSON SYSTEM

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Abstract

In this note we improve the results presented previously on global existence and global nonexistence for the solutions of the purely elliptic generalized Davey-Stewartson system. These results left a gap in the parameter range where neither a global existence result nor a global nonexistence result could be established. Here we are able to show when the coupling parameter is negative there is no gap. Moreover, in the case where the coupling parameter is positive we reduce the size of the gap.

Key words: Davey-Stewartson system, Global existence, Blow-up, Nonlinear Schrödinger equation.

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1. Introduction

In this short note we mainly improve the results obtained in [1] on the global existence and global nonexistence of solutions for the Cauchy problem of the generalized Davey-Stewartson (GDS) system. This system was derived in [2] and we refer the reader to that paper for the physical significance and for some special solutions of the GDS system. The conserved quantities, corresponding to conservation of mass and conservation of energy, play an important role in the study of the time asymptotic behaviour of this system. The blow-up of solutions for nonlinear Schrödinger (NLS) like systems usually follows from the virial identity combined with the existence of a solution with negative initial energy (see [3] and [4]). These conserved quantities and the virial identity for the GDS system were derived in [1]. Further conserved quantities, corresponding to scale invariance and pseudo-conformal invariance of solutions, have been discussed in [5]. It is worth mentioning that a certain amount of regularity is needed for the solutions in order for the above mentioned quantities to make sense, e.g. for the energy the solutions must be in the Sobolev space $H^1(\mathbb{R}^2)$, whereas for the virial identity one needs the solutions to be in the space Σ (see Lemma 1).

The results in [1] on the purely elliptic GDS system were two-fold: for some parameter range it was shown that the solutions exist globally in time whereas for a different parameter range there are solutions that can blow-up in finite time. Unfortunately, the two ranges on the parameters left a gap where neither a global existence nor a blow-up result was established. However, in the DS system, which is a close relative of the GDS system, there is no gap in the parameter ranges for the global existence and nonexistence of solutions (see [6]). It is therefore natural to ask whether one can close the gap in the GDS system as well. During some numerical experimentations on the blow-up solutions of the GDS system (see [7]), it is observed that under some parameter restrictions it should be possible to close the gap. In this note, we are able to close the gap analytically when the coupling parameter γ (see (1)) is negative and can only decrease the size of the gap when γ is positive.

Although the global existence of solutions for the GDS system follows from the control of the H^1 norms of the solutions, the standard blow-up mechanism requires the use of the virial identity in conjuction with the existence of a solution with negative initial energy. Since there are some technical difficulties in finding such a solution, the task of finding such a solution is more challenging then it appears at first sight.

The main section of this note starts with a summary of results that were given in [1]. Then, we show how to improve these results by a more careful analysis. In our main results, Theorem 3 and Lemma 2, we are able to extend the parameter ranges where the global existence and the blow-up results hold. In the case $\gamma < 0$, these two results leave no gap, however, when $\gamma > 0$ there still remains a gap albeit a smaller one. We close this note with some remarks concerning further applications of our observations above. In particular, we are able to improve the range of parameters for which the main results in [5] and [8] are applicable.

2. Global Existence and Global Nonexistence of Solutions

2.1 Background

We consider the GDS system, proposed in [2] to model $2+1$ dimensional wave propagation in a bulk medium composed of elastic material with couple stresses,

$$i\, u_t + \sigma u_{xx} + u_{yy} = \kappa \mid u \mid^2 u + \gamma(\varphi_{1,x} + \varphi_{2,y})u \tag{1}$$

$$\varphi_{1,xx} + m_2\varphi_{1,yy} + n\varphi_{2,xy} = (\mid u \mid^2)_x \tag{2}$$

$$\lambda\varphi_{2,xx} + m_1\varphi_{2,yy} + n\varphi_{1,xy} = (\mid u \mid^2)_y \ , \tag{3}$$

where u is the scaled complex amplitude of the free short transverse wave mode and φ_1 and φ_2 are the scaled free long longitudinal and long transverse wave modes, respectively. The parameters $\sigma, \kappa, \gamma, m_1, m_2, \lambda$ and n are real constants and σ is normalized as $\sigma = 1$. The parametric relation $(\lambda - 1)(m_2 - m_1) = n^2$ followed from the structure of the physical constants and played a key role in the analysis of these equations. All the results presented in this paper are derived assuming this parametric relation is valid.

The conserved quantities, corresponding to mass and energy, were derived in [2] in the form

$$\mathcal{N} = \int_{\mathbb{R}^2} |\, u \,|^2 \,\mathrm{d}x\mathrm{d}y \tag{4}$$

$$\begin{aligned}\mathcal{H} &= \int_{\mathbb{R}^2} \{\sigma|u_x|^2 + |u_y|^2 + \frac{\kappa}{2}|u|^4 + \frac{\gamma}{2}[(\varphi_{1,x})^2 + m_2(\varphi_{1,y})^2 \\ &\quad +\lambda(\varphi_{2,x})^2 + m_1(\varphi_{2,y})^2 + n(\varphi_{1,y}\varphi_{2,x} + \varphi_{1,x}\varphi_{2,y})]\}\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{5}$$

respectively. The function spaces that the solution $(u, \varphi_1, \varphi_2)$ of the initial-value problem resides in make these conserved quantities mathematically meaningful as well.

In [1], these equations were classified according to the signs of $(\sigma, m_1, m_2, \lambda)$. Here we will consider the purely elliptic case only, that is, the elliptic-elliptic-elliptic case: (+,+,+,+). In the purely elliptic case there are two types of results in the literature depending on the spaces that solutions live in. For solutions in $H^1(\mathbb{R}^2)$, an existence and uniqueness result is indicated followed by a global existence result in [1], see Theorem 1 below. For solutions that remain in Σ a nonexistence result is given in [1], see Lemma 1 below, and also the asymptotic behavior of solutions is studied in [5]. More precisely, in [1], the global existence of the solution of GDS system in the purely elliptic case where m_1, m_2 and λ are positive was proved through the following theorem.

Theorem 1. (Theorem 5.1 of [1]) Suppose that $m_1 > 0$ and $\kappa \geq \max\{-\gamma \max(1, \frac{1}{m_1}), 0\}$, then the solutions of the GDS system (1)-(3) are global, i.e. $T_{\max} = \infty$.

Also in [1], through the following lemma, a virial type identity was derived for the solutions of the initial value problem that are in weighted space $\Sigma = \{u \in H^1(\mathbb{R}^2) : (x^2+y^2)^{1/2}u \in L^2(\mathbb{R}^2)\}$.

Lemma 1. (Lemma 6.2 of [1]) Suppose that $m_1 > 1$ and $\kappa < \min(-\frac{\gamma}{m_1}, 0)$, then there exists $u_0 \in H^1(\mathbb{R}^2)$ such that $\mathcal{H}(u_0) < 0$.

Using this identity a blow-up result was established in [1] for the purely elliptic case.

Theorem 2. (Theorem 6.1 of [1]) Let u be the solution of the Cauchy problem in $H^1(\mathbb{R}^2)$ for the GDS system (1)-(3) with initial value $u_0 \in H^1(\mathbb{R}^2)$ and $\nabla\varphi_1, \nabla\varphi_2 \in \mathbb{L}^2(\mathbb{R}^2)$. If one of the following conditions holds

$$i)\quad \mathcal{H}(u_0) < 0,$$

$$ii)\quad \mathcal{H}(u_0) = 0 \text{ and } \operatorname{Im}\int_{\mathbb{R}^2} (xu_0^* u_{0,x} + yu_0^* u_{0,y})\mathrm{d}x\mathrm{d}y < 0,$$

$$iii)\quad \mathcal{H}(u_0) > 0 \text{ and } -\operatorname{Im}\int_{\mathbb{R}^2} (xu_0^* u_{0,x} + yu_0^* u_{0,y})\mathrm{d}x\mathrm{d}y \geq \sqrt{\mathcal{H}(u_0)\mathcal{V}(u_0)},$$

where

$$\mathcal{V}(u) = \int_{\mathbb{R}^2} (x^2 + y^2)|u|^2 \mathrm{d}x\mathrm{d}y,$$

then there exists $T_{\max} < \infty$ such that

$$\lim_{t \to T_{\max}^-} (\|u_x\|_{L^2} + \|u_y\|_{L^2}) = \infty,$$

that is, if the solution exists for large enough finite time then it will blow up in finite time.

In summary, it was shown in [1] that the solutions of GDS system in the purely elliptic case will exist globally provided that the coefficients of the nonlinear terms satisfy certain conditions, i.e. $\kappa \geq \max\{-\gamma \max(1, \frac{1}{m_1}), 0\}$ when $m_1 > 0$. As a consequence of global existence, stability of zero solution of the GDS system had been established under the same condition. In [1], it was also shown that the solutions of the GDS system can not exist globally in the purely elliptic case using the virial theorem, when $\kappa < \min(-\frac{\gamma}{m_1}, 0)$ and $m_1 > 1$. As a consequence of this result the nontrivial ground state solutions of the GDS system were shown to be unstable.

Contrary to the DS system the global existence and nonexistence results given in [1], Theorem 1 and Lemma 1 in conjunction with Theorem 2 respectively, do not cover the whole parameter range for κ and γ. Assuming that $m_1 > 1$, both global existence and/or nonexistence of solutions remained an open problem for $\kappa \geq \min(-\frac{\gamma}{m_1}, 0)$ and $\kappa < \max(-\gamma, 0)$. In other words, in [1] there is a gap in κ values, namely $\min(-\frac{\gamma}{m_1}, 0) \leq \kappa < \max(-\gamma, 0)$, where no results were given.

2.2 Global Existence of Solutions

In this work, we first introduce a revised version of Theorem 1 and relate it to global existence of the solutions. As was the case in the previous subsection, from this point on we will assume that $\sigma = 1$, λ, m_1 and $m_2 > 0$, i.e. the purely elliptic case.

Theorem 3. Suppose that $\lambda > 1, \;\; m_2 > m_1 > 0, \;\; n > 0$ and let

$$\delta_1 := \frac{1 + m_1 - 2n}{\lambda m_1 + m_2} \; .$$

If

$$\text{(i)} \qquad \gamma > 0 \;\; \text{and} \;\; \kappa \geq -\gamma \min\{\delta_1, \frac{1}{m_1}, 1\}$$

or

$$\text{(ii)} \qquad \gamma < 0 \;\; \text{and} \;\; \kappa \geq -\gamma \max(\frac{1}{m_1}, 1) \; ,$$

then the solutions of the GDS system are global, i.e. $T_{\max} = \infty$.

Proof. The energy (Hamiltonian) $\mathcal{H}$, (5), of the solution u of the GDS system (1)-(3) was expressed in [1] as

$$\mathcal{H}(u) = \int (\xi_1^2 + \xi_2^2)|\hat{u}|^2 d\xi_1 \; d\xi_2 + \frac{1}{2} \int \; (\kappa + \gamma\alpha)|\hat{f}|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \tag{6}$$

where (ξ_1, ξ_2) is the dual variable of (x, y), $\hat{u}$ is the Fourier transform of u, $\hat{f}$ denotes the Fourier transform of $|u|^2$ and

$$\alpha = \alpha(\xi_1, \xi_2) = \frac{\lambda\xi_1^4 + (1 + m_1 - 2n)\xi_1^2\xi_2^2 + m_2\xi_2^4}{(\lambda\xi_1^2 + m_2\xi_2^2)(\xi_1^2 + m_1\xi_2^2)} \; . \tag{7}$$

It is then possible to show that

$$\alpha_m \leq \alpha(\xi_1, \xi_2) \leq \alpha_M, \tag{8}$$

where

$$\alpha_m := \min\{\delta_1, \frac{1}{m_1}, 1\}, \qquad \alpha_M := \max(\frac{1}{m_1}, 1) \; . \tag{9}$$

Thus, if

$$\gamma > 0 \quad \text{and} \quad \kappa \geq -\gamma \min\{\delta_1, \frac{1}{m_1}, 1\} \tag{10}$$

or

$$\gamma < 0 \quad \text{and} \quad \kappa \geq -\gamma \max(\frac{1}{m_1}, 1), \tag{11}$$

then $\kappa + \gamma\alpha \geq 0$. In such a case, the energy (6) is bounded below by

$$\mathcal{H}(u) \geq \int (\xi_1^2 + \xi_2^2)|\hat{u}|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 = \int_{\mathbb{R}^2} (|u_x|^2 + |u_y|^2) \mathrm{d}x\mathrm{d}y \ . \tag{12}$$

Thus, conservation of mass (4) and energy (5) lead to a uniform bound on the H^1 norm of u stating that the solution is global if it exists locally.

2.3 Global Nonexistence of Solutions

In this subsection we show that the solutions of GDS system (1)-(3) in the purely elliptic case can not exist globally in time. To this end, as in [1], we use the *method of moments* (or the virial theorem) that is the classical approach to determine whether a given initial wave will collapse into a singular point in a finite time, i.e. the wave amplitude blows up at this point. The method was first developed by Vlasov *et al.* [9] and applied to self-focussing phenomena in the NLS equation and to the DS equations by Ablowitz and Segur [10]. The method was extended to the GDS system in [1].

Lemma 2. Suppose that $\lambda > 1$, $m_2 > m_1 > 0$, $n > 0$ and let

$$\delta_2 := \frac{\sqrt{m_1}(1 + m_1 - 2n) + \sqrt{\lambda m_2}(1 + m_1)}{(m_1\sqrt{\lambda} + \sqrt{m_1 m_2})(\sqrt{m_2} + \sqrt{\lambda m_1})} \ .$$

If

(i) $$\gamma > 0 \quad \text{and} \quad \kappa < -\gamma \min\{\delta_2, \frac{1}{m_1}, 1\}$$

or

(ii) $$\gamma < 0 \quad \text{and} \quad \kappa < -\gamma \max(\frac{1}{m_1}, 1) \ ,$$

then there exists $u_0 \in H^1(\mathbb{R}^2)$ such that $\mathcal{H}(u_0) < 0$.

Proof. Let us consider the function $v_\mu(x,y) = \mu \exp(-\frac{x^2}{\beta_1^2} - \frac{y^2}{\beta_2^2})$ in $H^1(\mathbb{R}^2)$, where μ, β_1 and β_2 are taken as positive. After some long calculations, as in [1], we get

$$\mathcal{H}(v_\mu) = \frac{\pi(\beta_1^2 + \beta_2^2)}{2\beta_1\beta_2}\mu^2 + \frac{\pi\beta_1\beta_2}{8}(\kappa + \gamma J)\mu^4, \tag{13}$$

where

$$J(\beta_1,\beta_2) = \frac{(m_2 + \sqrt{\lambda m_1 m_2})\beta_1^2 + [\sqrt{m_1}(1-2n+m_1) + \sqrt{\lambda m_2}(1+m_1)]\beta_1\beta_2 + (\lambda m_1 + \sqrt{\lambda m_1 m_2})\beta_2^2}{(m_1\sqrt{\lambda} + \sqrt{m_1 m_2})[\sqrt{m_1 m_2}\beta_1^2 + (\sqrt{m_2} + \sqrt{\lambda m_1})\beta_1\beta_2 + \sqrt{\lambda}\beta_2^2]}. \tag{14}$$

It is possible to show that

$$\min\{\delta_2, \frac{1}{m_1}, 1\} \le J(\beta_1,\beta_2) \le \max(\frac{1}{m_1}, 1). \tag{15}$$

Thus, if

$$\gamma > 0 \quad \text{and} \quad \kappa < -\gamma \min\{\delta_2, \frac{1}{m_1}, 1\} \tag{16}$$

or

$$\gamma < 0 \quad \text{and} \quad \kappa < -\gamma \max(\frac{1}{m_1}, 1), \tag{17}$$

then by selecting the parameters β_1 and β_2 appropriately we obtain $\kappa + \gamma J < 0$. Then by suitably choosing the arbitrary positive μ, we conclude that there exists solutions with initial value equal to $v_\mu(x,y)$ that make the energy (13) negative.

2.4 Closing the Gap

It is possible to show that (see Appendix A) for $\lambda > 1$, $m_2 > m_1 > 0$, $n > 0$ we have

$$\delta_1 < \delta_2 \ . \tag{18}$$

For $\gamma < 0$ there is no gap.

For $\gamma > 0$ there is a gap which is given by

$$-\gamma \min\{\delta_2, \frac{1}{m_1}, 1\} \le \kappa < -\gamma \min\{\delta_1, \frac{1}{m_1}, 1\} \ . \tag{19}$$

Contrary to the cubic NLS equation and the DS system our global existence and nonexistence results, Theorem 3 and Lemma 2 in conjunction with Theorem 2 respectively, do not still cover the whole parameter range for κ and γ if $\gamma > 0$. In other words, assuming that $\gamma > 0$, for the interval given by (19) both global existence and/or nonexistence of solutions remain an open problem.

In order to compare the interval (19) with that obtained in [1] in terms of the size of the interval we assume that $m_1 \geq 1$. Then the above interval (19) becomes

$$-\gamma \min(\delta_2, \frac{1}{m_1}) \leq \kappa < -\gamma \min(\delta_1, \frac{1}{m_1}) \ . \tag{20}$$

We have the following cases:

(i) For $\dfrac{1}{m_1} < \delta_1$, there is no gap.

(ii) For $\delta_1 < \dfrac{1}{m_1} < \delta_2$, the interval (20) becomes

$$-\frac{\gamma}{m_1} \leq \kappa < -\gamma\delta_1. \tag{21}$$

(iii) For $\delta_2 < \dfrac{1}{m_1}$, the interval (20) becomes

$$-\gamma\delta_2 \leq \kappa < -\gamma\delta_1. \tag{22}$$

In the cases (ii) and (iii) the interval (20) is smaller than the interval $-\frac{\gamma}{m_1} \leq \kappa < 0$ obtained in [1].

3. Conclusions

The way to show the global existence of solutions passes from the control of the H^1-norm of the solutions, ideally independent of time. Here the conservation of energy and the conservation of mass play the leading roles since through them one can control the H^1-norm. In [1], the contribution from the nonlinear terms turned out to be positive hence the Hamiltonian dominated the L^2-norm of the gradient of the solutions, this fact combined with the conservation of the L^2-norms of the solutions resulted in the desired control of the H^1-norm.

In [8], a different parameter range was considered. There the contributions of the nonlinear terms were controlled by a Gagliardo-Nirenberg type inequality when the mass of the solution is smaller than the mass of the ground state. The analysis done on the symbol α above allows us to improve the result of Theorem 2.3 in [8] slightly as well when $\gamma < 0$, the improved theorem now reads as : when $\gamma < 0$ and $\kappa < \max(0, -\gamma\alpha_m)$ or when $\gamma > 0$ and $\kappa < -\gamma\alpha_M$ with α_m and α_M are as defined in (9) solutions with small enough mass exist globally in time. It is worth adding further that with the same parameter restrictions, the proof for the existence of standing waves still goes through (see Theorem 2.2 in [8]) since the denominator of the functional $J(u)$ remains strictly negative for all u in H^1.

Due to the improved estimates of $\kappa + \gamma\alpha$ one can also deduce following [5] that whenever

$$\gamma > 0 \quad \text{and} \quad \kappa \geq -\gamma \min\{\delta_1, \frac{1}{m_1}, 1\} \tag{23}$$

or

$$\gamma < 0 \quad \text{and} \quad \kappa \geq -\gamma \max(\frac{1}{m_1}, 1), \tag{24}$$

the solutions of the GDS system that remain in Σ for all time have their L^p-norms go to zero as $t \to \infty$, for $p > 2$.

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References

[1] C. Babaoglu, A. Eden, S. Erbay, Global existence and nonexistence results for a generalized Davey-Stewartson system, J. Phys. A: Math. Gen. 37 (2004) 11531-11546.

[2] C. Babaoglu, S. Erbay, Two-dimensional wave packets in an elastic solid with couple stresses, Int. J. Non-Linear Mech. 39 (2004) 941-949.

[3] T. Cazenave, Semilinear Schrödinger Equations, American Mathematical Society (AMS) Courant Lecture Notes in Mathematics 10, New York, 2003.

[4] C. Sulem, P. Sulem, The Nonlinear Schrödinger Equation Self-Focusing and Wave Collapse, Springer-Verlag, Toronto, 1999.

[5] A. Eden, H.A. Erbay, G.M. Muslu, Two remarks on a generalized Davey-Stewartson system, Nonlinear Analysis TMA, 64 (2006) 979-986.

[6] J. M. Ghidaglia, J.C. Saut, On the initial-value problem for the Davey-Stewartson systems, Nonlinearity 3 (1990) 475-506.

[7] G.M. Muslu, Blow-up Solutions of the Generalized Davey-Stewartson System: Analytical and Numerical Results (Ph.D. Thesis), Istanbul Technical University, Istanbul, 2006.

[8] A. Eden, S. Erbay, Standing waves for a generalized Davey-Stewartson system, J. Phys. A: Math. Gen. 39 (2006) 13435-13444.

[9] S.N. Vlasov, V.A. Petrishchev, V.I. Talanov, Averaged description of wave beams in linear and nonlinear media (the method of moments), Radiophys. and Quantum Electronics 14 (1974) 1062-1070.

[10] M.J. Ablowitz, H. Segur, On the evolution of packets of water waves J. Fluid Mech. 92 (1979) 691-715.

Appendix A

From $\lambda > 1$, $m_2 > m_1 > 0$, and $n > 0$ it follows that

$$(1 + m_1 - 2n)(2m_1) < (1 + m_1)(\lambda m_1 + m_2) \ . \tag{A1}$$

Multiplying both sides by $\sqrt{\lambda m_2 / m_1}$ we get

$$2(1 + m_1 - 2n)\sqrt{\lambda m_1 m_2} < \sqrt{\frac{\lambda m_2}{m_1}}(1 + m_1)(\lambda m_1 + m_2) \ . \tag{A2}$$

The left hand side of (A2) is equal to

$$(1 + m_1 - 2n)[(\sqrt{\lambda m_1} + \sqrt{m_2})^2 - (\lambda m_1 + m_2)], \tag{A3}$$

substituting this into (A2) and rearranging a bit we have

$$(1 + m_1 - 2n)(\sqrt{\lambda m_1} + \sqrt{m_2})^2 < [(1 + m_1 - 2n) + \sqrt{\frac{\lambda m_2}{m_1}}(1 + m_1)](\lambda m_1 + m_2) \ . \tag{A4}$$

Dividing both sides by $(\sqrt{\lambda m_1} + \sqrt{m_2})^2(\lambda m_1 + m_2)$, it follows that

$$\frac{1 + m_1 - 2n}{\lambda m_1 + m_2} < \frac{\sqrt{m_1}(1 + m_1 - 2n) + \sqrt{\lambda m_2}(1 + m_1)}{(m_1\sqrt{\lambda} + \sqrt{m_1 m_2})(\sqrt{m_2} + \sqrt{\lambda m_1})} \tag{A5}$$

where the left hand side of (A5) is equal to δ_1 and right hand side is equal to δ_2.