SOME ASPECTS OF FINITE AMPLITUDE TRANSVERSE WAVES IN A COMPRESSIBLE HYPERELASTIC SOLID

by J. B. HADDOW

(*Department of Mechanical Engineering, University of Victoria, Victoria, BC, Canada* V8W 3P6)

and H. A. ERBAY

(*Department of Mathematics, Istanbul Technical University, Maslak* 80626*, Istanbul, Turkey*)

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Summary

Conditions are obtained on the strain-energy functions of compressible isotropic homogeneous hyperelastic solids which admit the propagation of finite amplitude transverse waves, without accompanying coupled longitudinal waves, or which admit the simultaneous propagation of uncoupled transverse and longitudinal waves. Three types of transverse waves are considered: plane shear waves, axisymmetric anti-plane shear waves and azimuthal shear waves. Static axisymmetric anti-plane shear and static azimuthal shear, which result in isochoric deformation, are also considered for comparison with the corresponding wave propagation problems.

1. Introduction

In this paper the terms pure transverse waves and pure longitudinal waves are used to describe waves which are linearly polarized in directions normal and parallel, respectively, to the direction of propagation. The paper is concerned with conditions on the strain-energy functions which admit the propagation of finite amplitude plane, axisymmetric anti-plane and azimuthal pure transverse waves in an isotropic homogeneous compressible hyperelastic solid. For conciseness, the term axial shear wave is used, rather than axisymmetric anti-plane wave. Since finite amplitude transverse waves cannot, in general, propagate in a compressible solid without coupled longitudinal waves, longitudinal waves are also considered.

Propagation of finite amplitude plane transverse waves in an elastic solid has been extensively studied, especially for incompressible solids; however, the studies of finite amplitude axial and azimuthal shear waves in compressible elastic solids are less extensive. An extensive treatment of the propagation of plane waves in a compressible hyperelastic solid has been given by John (**1**), who uses the terms strictly transverse and strictly longitudinal, rather than pure longitudinal or pure transverse. Plane longitudinal and transverse finite amplitude waves in various compressible elastic solids have also been considered by Currie and Hayes (**2**). Finite amplitude plane pure longitudinal waves can propagate in all directions in an initially underformed compressible isotropic hyperelastic solid and it has been shown by John (**1**), that this is also true for a Hadamard material, subjected to an initial homogeneous deformation. Also, for any compressible isotropic hyperelastic solid, plane pure longitudinal waves can propagate in the principal directions of strain of an initial homogeneous deformation.

Haddow and Tait (**3**) have proposed an extensive class of strain-energy functions for isotropic hyperelastic solids which admit propagation of finite amplitude plane pure transverse waves, and a subclass that admits propagation of finite amplitude pure axial shear waves. The propagation of finite amplitude azimuthal shear waves, in a compressible isotropic hyperelastic solid, is investigated by Haddow and Jiang (**4**) who conjecture that pure azimuthal shear waves cannot exist.

Conditions on strain-energy functions for compressible isotropic hyperelastic solids which admit finite static anti-plane pure shear have been obtained by Jiang and Beatty (**5**), Jiang and Ogden (**6**) and others cited in (**5**, **6**). Corresponding conditions for finite static azimuthal pure shear have been obtained by Polignone and Horgan (**7**), Beatty and Jiang (**8**) and Jiang and Ogden (**9**). The term static pure shear refers to static shearing deformation which results in isochoric deformation, in the absence of body forces.

In the present paper we consider plane wave propagation in the direction of a principal axis of initial homogeneous strain. Plane wave propagation is considered as a preliminary to the consideration of axial and azimuthal shear waves, in an initially undeformed isotropic hyperelastic solid, and to illustrate the possibility that a pure transverse wave can propagate in a solid which does not admit the simultaneous propagation of uncoupled pure longitudinal and pure transverse waves.

Conditions on the strain-energy function of a compressible hyperelastic solid, which admits propagation of finite amplitude pure transverse waves without coupled longitudinal waves, are obtained for plane transverse waves, axial shear waves, and axisymmetric azimuthal shear waves. These transverse waves are pure transverse so that the deformation, resulting from the wave propagation, is isochoric. Also obtained are conditions on the strain-energy function of a compressible isotropic hyperelastic solid which admits simultaneous propagation of uncoupled finite amplitude plane pure transverse and pure longitudinal waves. The theory presented neglects thermal effects, which is equivalent to assuming that the solid is piezotropic; consequently isothermal strain-energy functions are assumed to be valid for wave propagation.

Several classes of strain-energy functions, of interest in nonlinear elasticity theory, are used to illustrate application of the conditions obtained for propagation of pure transverse waves and simultaneous propagation of uncoupled finite amplitude pure transverse and pure longitudinal waves.

2. Plane waves

In this section we obtain conditions on the strain-energy function of an isotropic compressible hyperelastic solid which admits a finite amplitude plane pure transverse wave. In order to make the conditions more general, it is assumed that the solid has been subjected to an initial homogeneous deformation with constant principal stretches λ_1, λ_2 and λ_3 in the X_1, X_2 and X_3 directions, respectively, of a rectangular Cartesian coordinate system. A Lagrangian approach is used for which $x_i, i \in \{1, 2, 3\}$, and $X_\alpha, \alpha \in \{1, 2, 3\}$, denote the coordinates of a material particle in the spatial and natural reference configurations, respectively, referred to the same rectangular coordinate system. The initial homogeneous deformation is given by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3. \tag{2.1}$$

A superimposed pure transverse wave propagating in the X_1 direction and linearly polarized in the X_3 direction results in a time dependent deformation given by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3 + w(X_1, t), \tag{2.2}$$

where t is time and w is displacement due to the wave. Components of the deformation gradient tensor $\mathbf{F}$ and the right Cauchy–Green tensor, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, corresponding to (2.2), are

$$[\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \gamma & 0 & \lambda_3 \end{bmatrix}, \quad [\mathbf{C}] = \begin{bmatrix} \gamma^2 + \lambda_1^2 & 0 & \gamma\lambda_3 \\ 0 & \lambda_2^2 & 0 \\ \gamma\lambda_3 & 0 & \lambda_3^2 \end{bmatrix}, \tag{2.3}$$

where a superposed T denotes the transpose and $\gamma(X_1, t) = \partial w(X_1, t)/\partial X_1$. The basic invariants of $\mathbf{C}$ are

$$I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \tfrac{1}{2}(I_1^2 - \operatorname{tr}\mathbf{C}^2), \quad I_3 = \det\mathbf{C}, \tag{2.4}$$

where tr and det denote the trace and determinant, respectively, and it follows from (2.3) and (2.4) that

$$I_1 = \gamma^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \gamma^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2, \quad I_3 = (\lambda_1\lambda_2\lambda_3)^2. \tag{2.5}$$

The strain-energy function of an isotropic hyperelastic solid can be expressed as a function $W(I_1, I_2, I_3)$ of the basic invariants of $\mathbf{C}$ or as a function $\hat{W}(\mathbf{F}) = W(I_1(\mathbf{F}), I_2(\mathbf{F}), I_3(\mathbf{F}))$ of $\mathbf{F}$. For a compressible hyperelastic solid, $\mathbf{S} = \partial\hat{W}(\mathbf{F})/\partial\mathbf{F}$, where $\mathbf{S}$ is the nominal stress tensor (**10**), and, if the solid is isotropic,

$$\mathbf{S} = 2W_1\mathbf{F}^T + 2W_2(I_1\mathbf{F}^T - \mathbf{C}\mathbf{F}^T) + 2W_3 I_3\mathbf{F}^{-1}, \tag{2.6}$$

where $W_i = \partial W/\partial I_i$, $i \in \{1, 2, 3\}$. The strain energy and stress vanish in the natural reference configuration; consequently $W(3, 3, 1) = 0$ and $W_1(3, 3, 1) + 2W_2(3, 3, 1) + W_3(3, 3, 1) = 0$. These conditions are henceforth described as the necessary conditions for a compressible isotropic strain-energy function.

It may be deduced from (2.3) and (2.6) that the non-zero components of the nominal stress tensor are given by

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & 0 & S_{13} \\ 0 & S_{22} & 0 \\ S_{31} & 0 & S_{33} \end{bmatrix}.$$

The stress components we need to consider are S_{11} and S_{13}, and it follows from (2.3) and (2.6) that

$$S_{11} = 2\lambda_1 W_1 + 2\lambda_1(\lambda_2^2 + \lambda_3^2)W_2 + 2\lambda_1\lambda_2^2\lambda_3^2 W_3, \tag{2.7}$$

$$S_{13} = 2(W_1 + \lambda_2^2 W_2)\gamma, \tag{2.8}$$

where W_i, $i \in \{1, 2, 3\}$, are obtained in terms of $\lambda_1, \lambda_2, \lambda_3$ and $\gamma(X_1, t)$ by using (2.5).

A compressible isotropic hyperelastic solid admits the deformation (2.2) if its strain-energy function $W(I_1, I_2, I_3)$ satisfies certain conditions; these conditions are now determined. It follows from (2.3), (2.5) and (2.6) that I_1, I_2, I_3 and $\mathbf{S}$ can be expressed as functions of X_1 and t, since $\gamma = \gamma(X_1, t)$ and λ_1, λ_2 and λ_3 are constants. The non-trivial equations of motion are

$$\frac{\partial S_{11}}{\partial X_1} = 0, \quad \frac{\partial S_{13}}{\partial X_1} = \rho_0\frac{\partial V_3}{\partial t}, \tag{2.9}$$

where ρ_0 is the density in the natural reference configuration, and $V_3 = \partial w/\partial t$ is the component of the particle velocity in the X_3 direction. Equation $(2.9)_1$ is a necessary condition for the deformation (2.2) to be possible but a sufficient condition is $S_{11} = S_{11}^{\circ}$ for all $\gamma \in (-\infty, \infty)$, where S_{11}° is the constant value of S_{11}, corresponding to the homogeneous initial deformation (2.1). It follows from (2.7) that this condition can be put in the form,

$$2\lambda_1 W_1 + 2\lambda_1(\lambda_2^2 + \lambda_3^2)W_2 + 2\lambda_1\lambda_2^2\lambda_3^2 W_3 = S_{11}^{\circ} \quad \forall\gamma \in (-\infty, \infty), \tag{2.10}$$

where W_i, $i \in \{1, 2, 3\}$, are evaluated using (2.5), and S_{11}° is evaluated using (2.5) with $\gamma = 0$. For the special case $\lambda_1 = \lambda_2 = \lambda_3 = 1$, the condtion (2.10) becomes

$$W_1 + 2W_2 + W_3 = 0 \quad \forall\gamma \in (-\infty, \infty), \tag{2.11}$$

where $W_i = W_i(3 + \gamma^2, 3 + \gamma^2, 1)$, $i \in \{1, 2, 3\}$.

It may be deduced from $(2.9)_2$ and $\gamma = \partial w/\partial X_1$ that, if condition (2.10) is satisfied,

$$\left(\frac{1}{\rho_0}\frac{dS_{13}}{d\gamma}\right)\frac{\partial^2 w}{\partial X_1^2} = \frac{\partial^2 w}{\partial t^2}, \tag{2.12}$$

where S_{13} is given by (2.8). Equation (2.12) is hyperbolic and is a wave equation with wave speed $c(\gamma) = \sqrt{(1/\rho_0)\, dS_{13}/d\gamma}$ if $dS_{13}/d\gamma > 0$ for all $\gamma \in (-\infty, \infty)$. It then follows that a pure transverse wave is admitted if and only if (2.10) is satisfied and (2.12) is hyperbolic.

We now obtain conditions for the simultaneous propagation, in the X_1 direction, of a pure longitudinal wave, and a finite amplitude pure transverse wave linearly polarized in the X_3 direction. We assume that λ_2 and λ_3 remain fixed and replace $\lambda_1 X_1$ in (2.2) by $\lambda_1 X_1 + u(X_1, t)$, where u is the displacement in the X_1 direction, to obtain the deformation for simultaneous propagation of a shear wave and a longitudinal wave,

$$x_1 = \lambda_2 X_1 + u(X_1, t), \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3 + w(X_1, t). \tag{2.13}$$

In (2.3), (2.5), (2.7) and (2.8) λ_1 is replaced by $\hat{\lambda}_1(X_1, t) = \lambda_1 + \partial u/\partial X_1$. Since λ_2 and λ_3 remain fixed, the dependence of W and its derivatives on λ_2 and λ_3 is suppressed in the notation, and the strain-energy function is expressed as $W = \hat{W}(\hat{\lambda}_1, \gamma)$. The stress components, S_{11} and S_{13}, are then given by

$$\hat{S}_{11}(\hat{\lambda}_1, \gamma) = \partial\hat{W}(\hat{\lambda}_1, \gamma)/\partial\hat{\lambda}_1, \quad \hat{S}_{13}(\hat{\lambda}_1, \gamma) = \partial\hat{W}(\hat{\lambda}_1, \gamma)/\partial\gamma. \tag{2.14}$$

The non-trivial equations of motion are

$$\frac{\partial S_{11}}{\partial X_1} = \rho_0\frac{\partial V_1}{\partial t}, \quad \frac{\partial S_{13}}{\partial X_1} = \rho_0\frac{\partial V_3}{\partial t}, \tag{2.15}$$

where V_1 is the component of the particle velocity in the X_1 direction. It follows from (2.14), $V_1 = \partial u/\partial t$, $V_3 = \partial w/\partial t$, $\hat{\lambda}_1 = \lambda_1 + \partial u/\partial X_1$ and $\gamma = \partial w/\partial X_1$, that (2.15) can be put in the form

$$c_{11}\frac{\partial^2 u}{\partial X_1^2} + c_{12}\frac{\partial^2 w}{\partial X_1^2} = \frac{\partial^2 u}{\partial t^2}, \quad c_{21}\frac{\partial^2 u}{\partial X_1^2} + c_{22}\frac{\partial^2 w}{\partial X_1^2} = \frac{\partial^2 w}{\partial t^2}, \tag{2.16}$$

where $c_{11} = (1/\rho_0)\partial^2\hat{W}/\partial\hat{\lambda}_1^2$, $c_{12} = c_{21} = (1/\rho_0)\partial^2\hat{W}/\partial\hat{\lambda}_1\partial\gamma$ and $c_{22} = (1/\rho_0)\partial^2\hat{W}/\partial\gamma^2$. The system (2.16) is hyperbolic with real wave speeds if $\hat{W}(\hat{\lambda}_1, \gamma)$ is strictly convex, that is, if $c_{11} > 0$, $c_{11}c_{22} - c_{12}^2 > 0$ and $c_{22} > 0$. Furthermore if $c_{12} = 0$, c_{11} is a function of $\hat{\lambda}_1$ only, c_{22} is a function or γ only, and equations (2.16) are two uncoupled wave equations with wave speeds $\sqrt{c_{11}(\hat{\lambda}_1)}$ and $\sqrt{c_{22}(\gamma)}$ for the propagation of pure longitudinal and pure transverse waves, respectively. It then follows that the uncoupled simultaneous propagation of finite amplitude pure transverse and pure longitudinal waves is possible if and only if

$$c_{12} = \frac{1}{\rho_0}\frac{\partial^2 \hat{W}}{\partial\hat{\lambda}_1\partial\gamma} = 0, \quad c_{11} = \frac{1}{\rho_0}\frac{\partial^2 \hat{W}}{\partial\hat{\lambda}_1^2} > 0, \quad c_{22} = \frac{1}{\rho_0}\frac{\partial^2 \hat{W}}{\partial\gamma^2} > 0. \tag{2.17}$$

If $c_{12} \neq 0$, it can be deduced from (2.16) that, when a finite amplitude transverse wave and a coupled longitudinal wave propagate these waves are not pure transverse and pure longitudinal and are not linearly polarized. If c_{11}, c_{12} and c_{22} are constants, the waves are linearly polarized but are not pure transverse and pure longitudinal.

Particular strain-energy functions are now discussed and in what follows $H_i'(I_3) \equiv dH_i/dI_3$, $i \in \{1, 2, 3\}$, and μ, where $\mu > 0$, denotes the shear modulus for classical (linear) elasticity theory. The Hadamard strain-energy function can be put in the form

$$W = \frac{\mu}{2}[f(I_1 - 3) + (1 - f)(I_2 - 3) + H_3(I_3)], \tag{2.18}$$

where $f \in [0, 1]$ for compatibility with the classical theory and the necessary conditions are $H_3(1) = 0$ and $2 - f + H_3'(1) = 0$. The strain-energy function (2.18) satisfies condition (2.10) and (2.12) is hyperbolic since it follows from (2.5) and (2.8) that $dS_{13}/d\gamma = \mu(f + (1 - f)\lambda_2^2) > 0$ for all $f \in [0, 1]$ and all $\lambda_2 > 0$. Consequently (2.18) admits propagation of a pure transverse wave. It is easily shown that (2.18) satisfies $(2.17)_1$ and $(2.17)_3$. In (2.18), $H_3(I_3)$ is unspecified except for the necessary conditions $H_3(1) = 0$ and $2 - f + H_3'(1) = 0$; consequently c_{11} cannot be determined explicitly in terms of $\hat{\lambda}_1$ and γ unless a particular form is given for $H_3(I_3)$. However $c_{11} > 0$ if (2.18) satisfies the elliptic condition $\partial S_{11}/\partial\hat{\lambda}_1 > 0$; then (2.18) admits simultaneous uncoupled propagation of a finite amplitude pure transverse wave and a pure longitudinal wave.

A generalized Hadamard strain-energy function is of the form

$$W = \frac{\mu}{2}[H_1(I_3)(I_1 - 3) + H_2(I_3)(I_2 - 3) + H_3(I_3)], \tag{2.19}$$

where $H_3(1) = 0$ and $H_1(1) + 2H_2(1) + H_3'(1) = 0$ are the necessary conditions, and $H_1(1) + H_2(1) = 1$, for compatibility with classical elasticity theory. In general, (2.19) does not satisfy (2.10). However, (2.10) is satisfied by (2.19) if $H_1'(1) = 0$, $H_2'(1) = 0$ and the initial deformation is isochoric; that is, $\lambda_1\lambda_2\lambda_3 = 1$. It follows from (2.5) and (2.8) that $dS_{13}/d\gamma = \mu[H_1(1) + \lambda_2^2 H_2(1)]$; consequently (2.12) is hyperbolic and (2.19) admits the propagation of a pure transverse wave if $[H_1(1) + \lambda_2^2 H_2(1)] > 0$ in addition to the above conditions. However (2.19) does not admit the simultaneous propagation of uncoupled pure transverse and pure longitudinal waves since $(2.17)_1$ is not satisfied unless H_1 and H_2 are constants which gives the subclass (2.18).

An interesting class of strain-energy functions is of the form

$$W = \frac{\mu}{2}[f(I_1 - 3I_3^{1/3}) + (1 - f)(I_2/I_3 - 3I_3^{-1/3}) + H_3(I_3)], \tag{2.20}$$

where $H_3(1) = 0$ and $H_3'(1) = 0$ are the necessary conditions. The part $\frac{1}{2}\mu[H_3(I_3)]$ is zero for isochoric deformation and the part $\frac{1}{2}\mu[f(I_1 - 3I_3^{1/3}) + (1-f)(I_2/I_3 - 3I_3^{-1/3})]$ is zero for pure dilatation. The well-known Levinson and Burgess strain-energy function (**11**) given by

$$W = \frac{\mu}{2}\bigg[f(I_1 - 3) + (1-f)(I_2/I_3 - 1) + 2(1-2f)(I_3^{1/2} - 1) + \left(2f + \frac{4\nu - 1}{1-2\nu}\right)(I_3^{1/2} - 1)^2\bigg], \tag{2.21}$$

where $f \in [0, 1]$ and ν is Poisson's ratio for classical elasticity theory, is a particular example of the class (2.20). Condition (2.10) is not satisfied by (2.20), and the particular case (2.21), unless $f = 1$ which gives a subclass of (2.20) which can be put in the form

$$W = \frac{\mu}{2}[(I_1 - 3) + 3(1 - I_3^{1/3}) + H_3(I_3)]. \tag{2.22}$$

Since (2.22) is also a subclass of (2.18) we conclude that it admits propagation of finite amplitude pure transverse waves and uncoupled finite amplitude pure transverse and pure longitudinal waves if $H_3(I_3)$ is such that $c_{11} > 0$. The particular case of (2.21) with $f = 1$ satisfies the condition $c_{11} > 0$. It should be noted that the $H_3(I_3)$ in (2.20) and (2.22) is not the same function as in (2.18).

3. Axial and azimuthal shear waves

Cylindrical polar coordinates of a material point in the spatial and natural reference configurations are denoted by, (r, θ, z) and (R, Θ, Z), respectively, and we consider an initially undeformed, unbounded region, $R \geqslant A$, where A is a positive constant. Again we consider an isotropic compressible hyperelastic solid.

3.1 *Axial shear wave problem*

The deformation for the propagation of a pure axial shear wave is given by

$$r = R, \quad \theta = \Theta, \quad z = Z + w(R, t), \tag{3.1}$$

and the corresponding components of **F** and **C** are

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{bmatrix}, \quad [\mathbf{C}] = \begin{bmatrix} 1+\gamma^2 & 0 & \gamma \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{bmatrix}, \tag{3.2}$$

where $\gamma(R, t) = \partial w(R, t)/\partial R$. The invariants of **C** are

$$I_1 = I_2 = 3 + \gamma^2, \quad I_3 = 1. \tag{3.3}$$

It follows from (2.6) and (3.2) that the components of the nominal stress tensor are given by

$$[\mathbf{S}] = \begin{bmatrix} S_{Rr} & 0 & S_{Rz} \\ 0 & S_{\Theta\theta} & 0 \\ S_{Zr} & 0 & S_{Zz} \end{bmatrix}. \tag{3.4}$$

The stress components we need to consider are S_{Rr}, S_{Rz} and $S_{\Theta\theta}$ and it follows from (2.6) and (3.2) that

$$S_{Rr} = 2(W_1 + 2W_2 + W_3), \tag{3.5}$$

$$S_{Rz} = 2\gamma(W_1 + W_2), \tag{3.6}$$

$$S_{\Theta\theta} = 2[W_1 + (2+\gamma^2)W_2 + W_3], \tag{3.7}$$

where $W_i, i \in \{1, 2, 3\}$, are obtained in terms of $\gamma(R, t)$ using (3.3); consequently the stress components can be expressed as functions of R and t.

The conditions which a strain-energy function must satisfy if deformation (3.1) is possible are now determined. The non-trivial equations of motion are

$$\frac{\partial S_{Rr}}{\partial R} + \frac{S_{Rr} - S_{\Theta\theta}}{R} = 0, \tag{3.8}$$

$$\frac{\partial S_{Rz}}{\partial R} + \frac{S_{Rz}}{R} = \rho_0 \frac{\partial V_z}{\partial t}, \tag{3.9}$$

where $V_z = \partial w/\partial t$ is the axial component of velocity. Substituting (3.5) and (3.7) into (3.8) gives

$$R\frac{\partial}{\partial R}(W_1 + 2W_2 + W_3) = \gamma^2 W_2, \tag{3.10}$$

and substituting (3.6) into (3.9) gives

$$\frac{2}{R}\frac{\partial}{\partial R}[R\gamma(W_1 + W_2)] = \rho_0 \frac{\partial^2 w}{\partial t^2}, \tag{3.11}$$

where $W_i, i \in \{1, 2, 3\}$, are evaluated using (3.3). A finite amplitude pure transverse wave can propagate if the condition (3.10) is satisfied and the solution of (3.10) substituted in (3.11) results in a hyperbolic wave equation.

The static form of (3.10),

$$R\frac{d}{dR}(W_1 + 2W_2 + W_3) = \gamma^2 W_2, \tag{3.12}$$

has been given by Jiang and Beatty (**5**), and the static form of (3.11) is

$$R\gamma(W_1 + W_2) = C, \tag{3.13}$$

where the constant C is determined by the boundary condition at $R = A$. A strain-energy function admits pure static axial shear if it satisfies (3.12) and (3.13). The subclass

$$W = \frac{\mu}{2}[H_1(I_3)(I_1 - 3) + H_3(I_3)] \tag{3.14}$$

of the generalized Hadamard material (2.19), where $H_1'(1) = 0$ in addition to the necessary conditions $H_3(1) = 0$ and $H_3'(1) + H_1(1) = 0$, and the condition $H_1(1) = 1$ for consistency with linear elasticity theory, is an example of a class of strain-energy functions which admits both static pure axial shear and propagation of pure axial shear waves. A subclass of (3.14) with $H_1(I_3) = 1$ is

$$W = \frac{\mu}{2}[(I_1 - 3) + H_3(I_3)]. \tag{3.15}$$

It has been shown by Haddow and Mioduchowski (**12**) and by others in later papers that strain-energy functions (3.14) and (3.15) admit static pure axial shear. It is easily shown that (3.12) is satisfied by (3.14) and (3.15) for all $\gamma \in (-\infty, \infty)$, and (3.13) is satisfied by $\gamma = C/R$, where C is a constant; consequently (3.14) and (3.15) admit static pure axial shear. Also (3.10) is satisfied by (3.14) and (3.15) for all $\gamma \in (-\infty, \infty)$. It follows that (3.14) and (3.15) admit pure axial shear wave propagation since the wave equation

$$\frac{\partial^2 w}{\partial R^2} + \frac{1}{R}\frac{\partial w}{\partial R} = \frac{\rho_0}{\mu}\frac{\partial^2 w}{\partial t^2},$$

with wave speed $\sqrt{\mu/\rho_0} > 0$, is obtained from (3.11) and $\gamma = \partial w/\partial R$.

We now consider a class of strain-energy functions which admit static pure axial shear but not the propagation of pure axial shear waves. Jiang and Beatty (**5**) have shown that 'a non-trivial axisymmetric anti-plane shear is controllable for every generalized Hadamard material in the subclass for which

$$2H_1'(1) + 2H_2'(1) + H_2(1) = 0\text{'}. \tag{3.16}$$

That is (2.19) admits static pure axial shear if (3.16) is satisfied in addition to the conditions $H_3(1) = 0$, $H_1(1)+2H_2(1)+H_3'(1) = 0$ and $H_1(1)+H_2(1) = 1$. The strain-energy function (2.19), with condition (3.16), satisfies (3.12) if and only if $\gamma R = C$, where C is a constant, and (3.13) is then satisfied since $2(W_1 + W_2)/\mu = H_1(1) + H_2(1) = 1$. However, (2.19), with condition (3.16), satisfies (3.10) if and only if $\gamma R = \phi(t)$, where $\phi(t)$ is an arbitrary function; then (3.11) can only be satisfied if $\partial^2 w/\partial t^2 = 0$. It follows that (2.19) with (3.16) admits static pure axial shear, but does not admit the propagation of pure axial shear waves except for the subclass (3.14).

It may be deduced from (3.10) to (3.13) that a necessary but not sufficient condition for the propagation of a pure axial shear wave to be admitted is that static pure axial shear is admitted, or a sufficient but not necessary condition for static pure axial shear to be admitted is that propagation of a pure axial shear wave is admitted.

We now consider the possibility of the simultaneous propagation, in the R direction, of a pure longitudinal wave and a finite amplitude pure axial transverse wave. The deformation for simultaneous propagation of an axial transverse wave and a radial longitudinal wave is given by

$$r = r(R, t), \quad \theta = \Theta, \quad z = Z + w(R, t), \tag{3.17}$$

and the corresponding components of **F** and **C** are

$$[\mathbf{F}] = \begin{bmatrix} \delta & 0 & 0 \\ 0 & \lambda & 0 \\ \gamma & 0 & 1 \end{bmatrix}, \quad [\mathbf{C}] = \begin{bmatrix} \delta^2 + \gamma^2 & 0 & \gamma \\ 0 & \lambda^2 & 0 \\ \gamma & 0 & 1 \end{bmatrix}, \tag{3.18}$$

where $\delta = \partial r/\partial R$, $\gamma = \partial w/\partial R$ and $\lambda = r/R$. The invariants of **C** are

$$I_1 = 1 + \gamma^2 + \delta^2 + \lambda^2, \quad I_2 = \gamma^2\lambda^2 + \lambda^2 + \delta^2 + \delta^2\lambda^2, \quad I_3 = \delta^2\lambda^2. \tag{3.19}$$

It follows from (2.6) and (3.18) that the non-zero stress components are again given in (3.4) and we need to consider S_{Rr}, S_{Rz} and $S_{\Theta\theta}$. When the strain-energy function is expressed in the form $W = \hat{W}(\delta, \gamma, \lambda)$ the stress components, S_{Rr}, S_{Rz} and $S_{\Theta\theta}$, are given by

$$\hat{S}_{Rr}(\delta, \gamma, \lambda) = \frac{\partial \hat{W}}{\partial \delta}, \quad \hat{S}_{Rz}(\delta, \gamma, \lambda) = \frac{\partial \hat{W}}{\partial \gamma}, \quad \hat{S}_{\Theta\theta}(\delta, \gamma, \lambda) = \frac{\partial \hat{W}}{\partial \lambda}. \tag{3.20}$$

The non-trivial equations of motion are

$$\frac{\partial S_{Rr}}{\partial R} + \frac{S_{Rr} - S_{\Theta\theta}}{R} = \rho_0 \frac{\partial V_r}{\partial t}, \tag{3.21}$$

$$\frac{\partial S_{Rz}}{\partial R} + \frac{S_{Rz}}{R} = \rho_0 \frac{\partial V_z}{\partial t}, \tag{3.22}$$

where $V_r = \partial r/\partial t$ and $V_z = \partial w/\partial t$. Substitution of (3.20) into (3.21) and (3.22) gives

$$\frac{\partial^2 \hat{W}}{\partial \delta^2}\frac{\partial \delta}{\partial R} + \frac{\partial^2 \hat{W}}{\partial \delta \partial \gamma}\frac{\partial \gamma}{\partial R} + \frac{\partial^2 \hat{W}}{\partial \delta \partial \lambda}\frac{\partial \lambda}{\partial R} + \frac{1}{R}\left(\frac{\partial \hat{W}}{\partial \delta} - \frac{\partial \hat{W}}{\partial \lambda}\right) = \rho_0 \frac{\partial V_r}{\partial t}, \tag{3.23}$$

$$\frac{\partial^2 \hat{W}}{\partial \delta \partial \gamma}\frac{\partial \delta}{\partial R} + \frac{\partial^2 \hat{W}}{\partial \gamma^2}\frac{\partial \gamma}{\partial R} + \frac{\partial^2 \hat{W}}{\partial \gamma \partial \lambda}\frac{\partial \lambda}{\partial R} + \frac{1}{R}\frac{\partial \hat{W}}{\partial \gamma} = \rho_0 \frac{\partial V_z}{\partial t}. \tag{3.24}$$

Equations (3.23) and (3.24) are a hyperbolic system if $\hat{W}(\delta, \gamma, \lambda)$ is convex in δ and γ, that is, if

$$\frac{\partial^2 \hat{W}}{\partial \delta^2} > 0, \quad \frac{\partial^2 \hat{W}}{\partial \delta^2}\frac{\partial^2 \hat{W}}{\partial \gamma^2} - \left(\frac{\partial^2 \hat{W}}{\partial \gamma \partial \delta}\right)^2 > 0, \quad \text{and } \frac{\partial^2 \hat{W}}{\partial \gamma^2} > 0.$$

Also (3.23) and (3.24) are uncoupled if

$$\frac{\partial^2 \hat{W}}{\partial \delta \partial \gamma} = 0, \quad \frac{\partial^2 \hat{W}}{\partial \lambda \partial \gamma} = 0 \tag{3.25}$$

so that γ does not appear in (3.23), and δ and λ do not appear in (3.24). Consequently (3.25), $\partial^2 \hat{W}/\partial \delta^2 > 0$ and $\partial^2 \hat{W}/\partial \gamma^2 > 0$ are the necessary and sufficient conditions for the simultaneous propagation of uncoupled finite amplitude pure axial shear waves and radial pure longitudinal waves. It follows from (3.19) that only strain-energy functions of the class (3.15) satisfy (3.25).

A class of strain-energy functions proposed by Haddow and Tait (**3**),

$$W = H_1(I_3)F(I_1 - 3) + H_3(I_3)\frac{\partial F}{\partial I_1}, \tag{3.26}$$

where $H_1(1) = 1$, $H_1'(1) = 0$, $H_3(1) = 0$, $H_3'(1) = -1$, $F(I_1 - 3)$ is a continuous function with continuous derivatives and $F(0) = 0$, satisfies (3.10). A further condition on F, $F'(\gamma^2) + 2\gamma^2 F''(\gamma^2) > 0$, where the primes denote differentiation with respect γ^2, is required so that $dS_{Rz}/d\gamma > 0$, where S_{Rz} is given by (3.6). A hyperbolic wave equation is then obtained from (3.11) and a pure axial shear wave can propagate. Strain-energy functions (3.14) and (3.15) are special cases of (3.26) with $F(I_1 - 3) = \frac{1}{2}\mu(I_1 - 3)$. Our conjecture is that (3.26) is the most general class of strain-energy function which admits the propagation of pure axial shear waves.

3.2 *Azimuthal shear wave problem*

The deformation for the propagation of a pure azimuthal shear wave is given by

$$r = R, \quad \theta = \Theta + g(R, t), \quad z = Z, \tag{3.27}$$

and the corresponding components of **F** and **C** are

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\mathbf{C}] = \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.28}$$

where $\gamma(R, t) = R\partial g/\partial R$. The invariants of **C** are

$$I_1 = I_2 = 3 + \gamma^2, \quad I_3 = 1. \tag{3.29}$$

It follows from (2.6) and (3.28) that the components of the nominal stress tensor are

$$[\mathbf{S}] = \begin{bmatrix} S_{Rr} & S_{R\theta} & 0 \\ S_{\Theta r} & S_{\Theta\theta} & 0 \\ 0 & 0 & S_{Zz} \end{bmatrix}.$$

The stress components we need to consider are S_{Rr}, $S_{R\theta}$ and $S_{\Theta\theta}$ and the following expressions are found from (2.6) and (3.28):

$$S_{Rr} = 2W_1 + 4W_2 + 2W_3, \tag{3.30}$$

$$S_{R\theta} = 2\gamma(W_1 + W_2), \tag{3.31}$$

$$S_{\Theta\theta} = 2W_1 + 4W_2 + 2W_3, \tag{3.32}$$

where W_i, $i \in \{1, 2, 3\}$, are obtained in terms of $\gamma(R, t)$ using (3.29). The non-trivial equations of motion are

$$\frac{\partial S_{Rr}}{\partial R} + \frac{S_{Rr} - S_{\Theta\theta}}{R} - \alpha S_{R\theta} = -\rho_0 \frac{V_\theta^2}{R}, \tag{3.33}$$

$$\frac{\partial S_{R\theta}}{\partial R} + \frac{2S_{R\theta}}{R} = \rho_0 \frac{\partial V_\theta}{\partial t}, \tag{3.34}$$

where $\alpha = \partial g/\partial R$, $V_\theta = R\partial g/\partial t$. Substituting (3.30) to (3.32) into (3.33) and (3.34) gives

$$2\frac{\partial}{\partial R}(W_1 + 2W_2 + W_3) - \frac{2\gamma^2}{R}(W_1 + W_2) = -\rho_0 \frac{V_\theta^2}{R}, \tag{3.35}$$

$$\frac{2}{R^2}\frac{\partial}{\partial R}[R^2\gamma(W_1 + W_2)] = \rho_0 \frac{\partial V_\theta}{\partial t}. \tag{3.36}$$

Unlike the corresponding equations (3.10) and (3.11) for the axial case, (3.35) and (3.36) both have an acceleration term on the right-hand side.

It has been shown in (**7** to **9**) that there are strain-energy functions for compressible isotropic hyperelastic solids which admit static pure azimuthal shear. These strain-energy functions satisfy the conditions

$$\frac{d}{dR}(W_1 + 2W_2 + W_3) - \frac{\gamma^2}{R}(W_1 + W_2) = 0, \tag{3.37}$$

$$\frac{d}{dR}[R^2\gamma(W_1 + W_2)] = 0, \tag{3.38}$$

which are the static forms of (3.35) and (3.36).

An example of a class of strain-energy functions which admits static finite pure azimuthal shear but does not admit the propagation of finite amplitude pure azimuthal shear waves is now discussed. A subclass of the generalized Hadamard strain-energy function (2.19) for which

$$H_1'(1) + H_2'(1) = -1/4 \tag{3.39}$$

satisfies (3.37) if and only if $\gamma R^2 = C$, where C is a constant, and (3.38) is then satisfied since $2(W_1 + W_2)/\mu = H_1(1) + H_2(1) = 1$. It follows that this subclass of (2.19) with (3.39) admits static pure azimuthal shear. It can be shown that it does not admit propagation of a pure azimuthal shear wave since conditions (3.35) and (3.36) give

$$\frac{1}{R^2}\left\{\frac{\partial(R^2\gamma)}{\partial R}\right\} = \frac{2V_\theta^2}{c^2\gamma R} \tag{3.40}$$

and

$$\frac{1}{R^2}\left\{\frac{\partial(R^2\gamma)}{\partial R}\right\} = \frac{1}{c^2}\frac{\partial V_\theta}{\partial t}, \tag{3.41}$$

respectively, where $c^2 = \mu/\rho_0$. Conditions (3.40) and (3.41) are satisfied only if $V_\theta \equiv 0$, and then the relation $R^2\gamma = C$ for static pure azimuthal shear is obtained. Consequently this subclass does not admit pure azimuthal shear waves. None of the other strain-energy functions, given in (**7** to **9**), which admit static pure azimuthal shear and none of the strain-energy functions we have investigated, satisfy (3.35) and (3.36). It has been noted by Haddow and Jiang (**4**) that, for the propagation of finite amplitude azimuthal shear waves in a hyperelastic solid, the governing equations cannot be separated into two uncoupled systems, which govern the propagation of pure azimuthal shear waves and pure radial longitudinal waves, respectively. This means that the simultaneous propagation of a finite amplitude pure azimuthal shear wave and a pure radial longitudinal wave is not admitted in any compressible isotropic hyperelastic solid. These considerations lead to our conjecture that there is no compressible isotropic hyperelastic solid which admits the propagation of a finite amplitude pure azimuthal shear wave; however, we have been unable to prove this.

4. Concluding remarks

The above discussion of wave propagation is based on a purely mechanical theory and neglects thermal effects. This is equivalent to assuming that the solid is piezotropic, that is, mechanical and thermal effects are uncoupled. A more sophisticated approach is to consider isentropic deformation which takes account of the thermodynamic equations of state of the solid but assumes that the solid is non-heat-conducting. This approach involves thermoelastic equations of state which have been given by Chadwick (**13**) and Chadwick and Creasy (**14**), for rubberlike materials. It has been shown (**15**) that use of the thermoelastic equations of state gives negligible differences from the purely mechanical theory for moderate finite dynamic shear deformation, say $\gamma < 1$, when mechanical and thermal properties, representative of rubberlike materials, are considered. Greater differences occur for longitudinal waves; however, for the problems considered in this paper we are mainly concerned with transverse waves.

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