INSTABILITY AND STABILITY PROPERTIES OF TRAVELING WAVES FOR THE DOUBLE DISPERSION EQUATION

H. A. ERBAY, S. ERBAY, AND A. ERKIP

Abstract. In this article we are concerned with the instability and stability properties of traveling wave solutions of the double dispersion equation $u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} = -(|u|^{p-1}u)_{xx}$ for $p > 1$, $a > b > 0$. The main characteristic of this equation is the existence of two sources of dispersion, characterized by the terms u_{xxxx} and u_{xxtt}. We obtain an explicit condition in terms of a, b and p on wave velocities ensuring that traveling wave solutions of the double dispersion equation are strongly unstable by blow up. In the special case of the Boussinesq equation ($b = 0$), our condition reduces to the one given in the literature. For the double dispersion equation, we also investigate orbital stability of traveling waves by considering the convexity of a scalar function. We provide analytical as well as numerical results on the variation of the stability region of wave velocities with a, b and p and then state explicitly the conditions under which the traveling waves are orbitally stable.

1. Introduction

The present paper is concerned with the instability and stability properties of traveling wave solutions for the double dispersion equation

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} = -(|u|^{p-1}u)_{xx}, \tag{1.1}$$

where a, b are positive real constants with $a > b$, and $p > 1$. In particular we prove that traveling wave solutions are unstable by blow-up if the wave velocities of the traveling waves are less than a critical wave velocity. We also state explicitly a set of conditions on a, b and p for which the traveling waves are orbitally stable.

The double dispersion equation (1.1) was derived as a mathematical model of the propagation of dispersive waves in a wide variety of situations, see for instance [1, 2] and the references therein. Well posedness (and related properties) of the Cauchy problem for the double dispersion equation have been studied in the literature by several authors [3, 4, 5]. It is interesting to note that (1.1) is a special case of the general class of nonlinear nonlocal wave equations

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}, \tag{1.2}$$

with pseudo-differential operators L and B, studied in [6, 7, 8]. Indeed, for the case

$$L = (I - aD_x^2)(I - bD_x^2)^{-1},\ B = (I - bD_x^2)^{-1},\ g(u) = -|u|^{p-1}u, \tag{1.3}$$

where I is the identity operator and D_x denotes the partial derivative with respective to x, (1.2) reduces to (1.1). The well-posedness of the Cauchy problem for the

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general class (1.2) was studied in [6] and then the parameter dependent thresholds for global existence versus blow-up were established in [7] for power nonlinearities. In a recent study [8] on (1.2), again for power nonlinearities, the existence of traveling wave solutions $u = \phi_c(x - ct)$, where $c \in \mathbb{R}$ is the wave velocity, has been established and orbital stability of the traveling waves has been studied. The orbital stability is based on the convexity of a certain function $d(c)$ related to conserved quantities. Furthermore, it has been shown that when $L = I$, (1.2) becomes a special case of the Klein-Gordon-type equations and $d(c)$ can be computed explicitly. In [8] the sharp threshold of instability/stability of traveling waves for this regularized Klein-Gordon equation has been established. In other words, for $L = I$, it has been shown that traveling wave solutions of (1.2) are orbitally stable for

$$\frac{p-1}{p+3} < c^2 < 1 \tag{1.4}$$

and are unstable by blow-up for

$$c^2 < \frac{p-1}{p+3}. \tag{1.5}$$

It remains an open question, however, whether a sharp threshold of instability/stability can be obtained for the double dispersion equation (1.1) which is another special case of (1.2).

For some limiting cases of (1.1), the above question was fully answered in the literature. For the special case $a = 1$, $b = 0$; (1.1) becomes the (generalized) Boussinesq equation [9]

$$u_{tt} - u_{xx} + u_{xxxx} = -(|u|^{p-1}u)_{xx} \tag{1.6}$$

which has received much attention in the literature. It was established in [10] that solitary wave solutions of (1.6) are orbitally stable if

$$\frac{p-1}{4} < c^2 < 1 \text{ and } 1 < p < 5. \tag{1.7}$$

In [11], it was proved that solitary waves for (1.6) are orbitally unstable if

$$c^2 < \frac{p-1}{4} \text{ and } 1 < p < 5, \tag{1.8}$$

or

$$c^2 < 1 \text{ and } p \geq 5. \tag{1.9}$$

On the other hand, in [12] it was shown that traveling wave solutions of (1.6) are strongly unstable by blow-up for

$$c^2 < \frac{p-1}{2(p+1)}. \tag{1.10}$$

In the limiting case $a = b$; (1.1) reduces to

$$u_{tt} - u_{xx} = -(1 - bD_x^2)^{-1}(|u|^{p-1}u)_{xx}, \tag{1.11}$$

which is a special case of the regularized Klein-Gordon equation studied in [8] and therefore the results given by (1.4) and (1.5) are also valid for this special case. For the special case $a = 0$, $b = 1$; (1.1) reduces to the improved Boussinesq equation [13]

$$u_{tt} - u_{xx} - u_{xxtt} = -(|u|^{p-1}u)_{xx}, \tag{1.12}$$

which has no traveling wave solution due to the minus sign on the right hand side. For the sake of completeness, we point out that, in [14], a sufficient condition for orbital stability of solitary waves was given for a more general version of (1.1):

$$(1 + \gamma |D_x|^\nu)\, u_{tt} - (a_0 + a_1 |D_x|^\nu)\, u_{xx} = -\Big(|u|^{p-1} u\Big)_{xx}, \tag{1.13}$$

where $\nu \ge 1$, $\gamma > 0$, a_0 and a_1 are real constants.

The aim of the present study is to investigate instability/stability properties of traveling wave solutions for (1.1) when $a > b > 0$. Our main result is that for all wave velocities c with $c^2 < c_0^2$ where

$$c_0^2 = \left(\frac{p-1}{p+1}\right)\left[1 + \left(1 - \frac{b(p+3)(p-1)}{a(p+1)^2}\right)^{1/2}\right]^{-1}, \tag{1.14}$$

traveling wave solutions of (1.1) are unstable by blow-up. It is important to note that our condition $c^2 < c_0^2$ for instability by blow-up matches the known results in the two limiting cases $a = 1$, $b = 0$ and $a = b$. That is, as it is expected, it reduces to (1.10) when $a = 1$, $b = 0$ and to (1.5) when $a = b$. For the other result of this work, we investigate both analytically and numerically orbital stability of traveling waves by applying the convexity criterion to (1.1). We then identify conditions (see (4.8)-(4.10)) on wave velocity and the parameters a, b and p for which traveling wave solutions of (1.1) are orbitally stable. Recalling that we restrict the discussion to the case $a > b > 0$, one may ask whether similar conclusions are still true if $a < b$. We emphasize that for $a < b$, (1.1) has traveling wave solutions with $c^2 < a/b$ but we cannot make a conclusion about instability by blow-up in this case. The crucial fact is that for $a < b$ the dispersive term u_{xxtt} in (1.1) dominates and thus (1.1) behaves much like (1.12). It seems that our restriction $a > b$ is more structural than a technical one.

The structure of the paper is as follows. In Section 2 we first review some previously known results, including the local existence theorem and the conserved quantities, for (1.1) and then discuss the Pohozaev identities and the invariant sets. In Section 3, we prove instability by blow-up of traveling waves with $c^2 < c_0^2$ for (1.1). In Section 4, we announce orbital stability conditions for traveling wave solutions of (1.1).

Throughout this paper, we use the standard notation for function spaces. The symbol $\widehat{u}$ represents the Fourier transform of u, defined by $\widehat{u}(\xi) = \int_{\mathbb{R}} u(x)e^{-i\xi x}dx$. The L^p $(1 \le p < \infty)$ norm of u on $\mathbb{R}$ is denoted by $\|u\|_{L^p}$. The inner product of u and v in L^2 is represented by $\langle u, v\rangle$. The L^2 Sobolev space of order s on $\mathbb{R}$ is denoted by $H^s = H^s(\mathbb{R})$ with the norm $\|u\|_{H^s}^2 = \int_{\mathbb{R}}(1+\xi^2)^s|\widehat{u}(\xi)|^2 d\xi$. The symbol $\mathbb{R}$ in $\int_{\mathbb{R}}$ will be mostly suppressed to simplify exposition.

2. Pohozaev Identities and Invariant Sets

2.1. Preliminaries: Local Existence and Conserved Quantities.

We now list some preliminary results for (1.1) (or equivalently, for (1.2) with (1.3)). Local existence of the Cauchy problem for (1.1) has been established in [3]. The local existence result given in [6] for (1.2) will also apply. For our purposes it is sufficient to consider solutions in H^1 and therefore we restrict our remarks concerning (1.1) to this case. The local existence result in [3] implies that for initial data in $H^1 \times L^2$, the Cauchy problem for (1.1) has a unique solution $u \in C([0,T), H^1) \cap C^1([0,T), L^2)$

for some $T>0$. As in [7], we now introduce new variables (u,w), where $u=v_x$ and $w=v_t$ for a suitable function v. Then we consider the following equivalent initial-value problem:

$$u_t = w_x,\ x\in\mathbb{R},\ t>0 \tag{2.1}$$

$$w_t = (1-bD_x^2)^{-1}\left[(1-aD_x^2)u_x - (|u|^{p-1}u)_x\right],\ x\in\mathbb{R},\ t>0 \tag{2.2}$$

$$u(x,0)=u_0(x),\ w(x,0)=w_0(x),\ x\in\mathbb{R} \tag{2.3}$$

for which the local existence theorem in [7] is rephrased as follows.

Theorem 2.1. *For initial data $U_0=(u_0,w_0)\in H^1\times H^1$, there exists some $T>0$ so that the Cauchy problem (2.1)-(2.3) is locally well-posed with solution $U=(u,w)\in C([0,T),H^1\times H^1)$.*

The energy and momentum functionals given in [7] turn out to be

$$E(U)=E(u,w)=\frac{1}{2}\int(w^2+bw_x^2)dx+\frac{1}{2}\int(u^2+au_x^2)dx-\frac{1}{p+1}\int|u|^{p+1}dx, \tag{2.4}$$

$$M(U)=M(u,w)=\int(uw+bu_xw_x)dx \tag{2.5}$$

for (2.1)-(2.2). The energy and momentum are conserved quantities of (2.1)-(2.2), namely for a solution $U(t)$ of (2.1)-(2.2) both $E(U(t))$ and $M(U(t))$ are independent of t [7]. We note that $H^1\times H^1$ is the natural energy and momentum space.

2.2. **Pohozaev Identities and Invariant Sets.**

Traveling wave solutions $u(x,t)=\phi_c(x-ct)$ of (1.1) satisfy the differential equation

$$(a-bc^2)\phi_c'' - (1-c^2)\phi_c + |\phi_c|^{p-1}\phi_c = 0 \tag{2.6}$$

where we have assumed that ϕ_c and all its derivatives decay at infinity. For $a-bc^2>0$ and $1-c^2>0$, (2.6) has a unique solution up to translation, namely

$$\phi_c(x)=\left[\frac{1}{2}(p+1)(1-c^2)\right]^{\frac{1}{p-1}}\operatorname{sech}^{\frac{2}{p-1}}\left[\frac{1}{2}(p-1)(\frac{1-c^2}{a-bc^2})^{\frac{1}{2}}x\right]. \tag{2.7}$$

As we assume that $a>b$, the above two conditions given for the wave velocity reduce to $c^2<\min\{1,a/b\}=1$. We note that this is exactly the bound obtained in [8], which is due to the fact that the symbol $l(\xi)$ of the operator L in (1.3) satisfies

$$1\le l(\xi)=\frac{1+a\xi^2}{1+b\xi^2}\le\frac{a}{b}$$

for $a>b$.

We make extensive use of the following two Pohozaev identities.

Lemma 2.2. *Traveling wave solutions of (2.6) satisfy the Pohozaev identities*

$$(1-c^2)\|\phi_c\|_{L^2}^2+(a-bc^2)\|\phi_c'\|_{L^2}^2-\|\phi_c\|_{L^{p+1}}^{p+1} = 0 \tag{2.8}$$

$$\frac{(1-c^2)}{2}\|\phi_c\|_{L^2}^2-\frac{(a-bc^2)}{2}\|\phi_c'\|_{L^2}^2-\frac{1}{p+1}\|\phi_c\|_{L^{p+1}}^{p+1} = 0. \tag{2.9}$$

Proof. The first identity is obtained multiplying (2.6) by ϕ_c and then integrating the resulting equation over $\mathbb{R}$. To obtain the second one we multiply (2.6) by $x\phi_c'$ and again integrate. ∎

To simplify the notation, from now on we will fix c with $c^2 < 1$ and let

$$A = 1 - c^2, \quad B = a - bc^2.$$

For $u \in H^1$ we define two functionals, P_1 and P_2, as follows:

$$P_1(u) = A\|u\|_{L^2}^2 + B\|u_x\|_{L^2}^2 - \|u\|_{L^{p+1}}^{p+1}, \tag{2.10}$$

$$P_2(u) = \frac{A}{2}\|u\|_{L^2}^2 - \frac{B}{2}\|u_x\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{2.11}$$

From Lemma 2.2 we have $P_1(\phi_c) = 0$ and $P_2(\phi_c) = 0$. Moreover, we note that $P_1(u)$ coincides with the functional $2\mathcal{I}_c(u) - \mathcal{Q}(u)$ of [8] (and with $2\mathcal{I}_\gamma(u) - \mathcal{Q}(u)$ of [7]). As in [7] and [8], using (2.4) and (2.5) we get the following identity:

$$E(u, w) + cM(u, w) = \frac{1}{2}\|w + cu\|_{L^2}^2 + \frac{b}{2}\|w_x + cu_x\|_{L^2}^2 + V(u), \tag{2.12}$$

where $V(u)$ is defined as

$$V(u) = \frac{A}{2}\|u\|_{L^2}^2 + \frac{B}{2}\|u_x\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}. \tag{2.13}$$

In what follows, for the traveling wave solution $u(x,t) = \phi_c(x - ct)$, the corresponding solution of (2.1)-(2.2) will be denoted by $U(x,t) = \Phi_c(x - ct)$ in which $\Phi_c(x) = (\phi_c(x), -c\phi_c(x))$. From (2.12) and (2.13) it follows that

$$E(\Phi_c) + cM(\Phi_c) = V(\phi_c). \tag{2.14}$$

We now rephrase Lemma 4.1 of [7] and Lemma 4.2 of [8] together as follows:

Lemma 2.3. $d(c) = \inf\left\{V(u) : u \in H^1,\ u \neq 0,\ P_1(u) = 0\right\}$ *is attained at the travelling wave* ϕ_c. *Moreover*

$$\inf\left\{E(U) + cM(U) : U = (u, w) \in H^1 \times H^1,\ u \neq 0,\ P_1(u) = 0\right\} = d(c).$$

For $\alpha \in \mathbb{R}$, we now define a functional, $K_\alpha(u)$, as follows:

$$\begin{aligned} K_\alpha(u) &= \alpha P_1(u) + P_2(u) \\ &= \frac{A}{2}(2\alpha + 1)\|u\|_{L^2}^2 + \frac{B}{2}(2\alpha - 1)\|u_x\|_{L^2}^2 - (\alpha + \frac{1}{p+1})\|u\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{2.15}$$

Note that $K_\alpha(\phi_c) = 0$ for all α. Consider the family of minimization problems

$$d_\alpha(c) = \inf\left\{V(u) : u \in H^1,\ u \neq 0,\ K_\alpha(u) = 0\right\}.$$

Following the scaling idea in [15], we prove:

Lemma 2.4. *For every* $\alpha > \frac{1}{2}$ *we have* $d_\alpha(c) = d(c)$.

Proof. Since $K_\alpha(\phi_c) = 0$ we have $d_\alpha(c) \leq V(\phi_c) = d(c)$. For the converse, we take some $u \neq 0$ with $K_\alpha(u) = 0$. If $P_1(u) = 0$, then by Lemma 2.3 we have $V(u) \geq d(c)$. We now turn to the case $P_1(u) \neq 0$. For $\lambda > 0$ we let $u_\lambda(x,t) = \lambda^\alpha u\left(\frac{x}{\lambda}, t\right)$. Substituting u_λ into (2.10) yields

$$\begin{aligned} P_1(u_\lambda) &= A\lambda^{2\alpha+1}\|u\|_{L^2}^2 + B\lambda^{2\alpha-1}\|u_x\|_{L^2}^2 - \lambda^{\alpha(p+1)+1}\|u\|_{L^{P+1}}^{p+1} \\ &= \lambda^{2\alpha-1}\left(A\lambda^2\|u\|_{L^2}^2 + B\|u_x\|_{L^2}^2 - \lambda^{\alpha(p-1)+2}\|u\|_{L^{P+1}}^{p+1}\right), \end{aligned}$$

from which it follows that $P_1(u_\lambda)$ is positive for small λ but negative for large λ. Hence there is some λ_0 for which $P_1(u_{\lambda_0}) = 0$. Thus, by Lemma 2.3, we have $V(u_{\lambda_0}) \geq d(c)$. On the other hand, computation of $V(u)$ at u_λ gives

$$V(u_\lambda) = \frac{A}{2}\lambda^{2\alpha+1}\|u\|_{L^2}^2 + \frac{B}{2}\lambda^{2\alpha-1}\|u_x\|_{L^2}^2 - \frac{1}{p+1}\lambda^{\alpha(p+1)+1}\|u\|_{L^{P+1}}^{p+1}.$$

Differentiating this we get

$$\begin{aligned}\frac{dV(u_\lambda)}{d\lambda} &= \frac{A}{2}(2\alpha+1)\lambda^{2\alpha}\|u\|_{L^2}^2 + \frac{B}{2}(2\alpha-1)\lambda^{2\alpha-2}\|u_x\|_{L^2}^2 \\ &\quad - \frac{\alpha(p+1)+1}{p+1}\lambda^{\alpha(p+1)}\|u\|_{L^{P+1}}^{p+1} \\ &= \lambda^{2\alpha-2}g(\lambda)\end{aligned}$$

with

$$g(\lambda) = \frac{A}{2}(2\alpha+1)\lambda^2\|u\|_{L^2}^2 + \frac{B}{2}(2\alpha-1)\|u_x\|_{L^2}^2 - \frac{\alpha(p+1)+1}{p+1}\lambda^{\alpha(p-1)+2}\|u\|_{L^{P+1}}^{p+1}.$$

It is easy to se that $g'(\lambda)$ changes sign from positive to negative exactly once on $(0, \infty)$. We observe that when $2\alpha - 1 > 0$, the function $g(\lambda)$ is positive for small λ but negative for large λ. Hence we conclude that $g(\lambda)$ changes its sign exactly once on $(0, \infty)$. The same conclusion holds for $\frac{d}{d\lambda}V(u_\lambda)$. This in turn shows that $V(u_\lambda)$ attains its global maximum at exactly one point in $(0, \infty)$. Moreover

$$\begin{aligned}\frac{dV(u_\lambda)}{d\lambda}\Big|_{\lambda=1} &= \frac{A}{2}(2\alpha+1)\|u\|_{L^2}^2 + \frac{B}{2}(2\alpha-1)\|u_x\|_{L^2}^2 - \frac{\alpha(p+1)+1}{p+1}\|u\|_{L^{P+1}}^{p+1} \\ &= K_\alpha(u) = 0,\end{aligned}$$

so that the maximum is attained at $\lambda = 1$. This means $V(u) \geq V(u_{\lambda_0}) \geq d(c)$. So we have $d_\alpha(c) \geq d(c)$. This completes the proof. □

We now let

$$\widetilde{\Sigma}_\alpha = \left\{U \in H^1 \times H^1 : E(U) + cM(U) < d(c),\ K_\alpha(u) < 0\right\}.$$

Lemma 2.5. *Let $\alpha > \frac{1}{2}$. Then $\widetilde{\Sigma}_\alpha$ is invariant under the flow defined by the Cauchy problem (2.1)-(2.3).*

Proof. Suppose $U_0 \in \widetilde{\Sigma}_\alpha$ and let $U(t)$ be the solution of (2.1)-(2.3) with initial value U_0. Since E and M are conserved quantities, then $E(U(t))+cM(U(t)) < d(c)$. Assume that $U(t)$ does not stay in $\widetilde{\Sigma}_\alpha$. Then there is some t_1 for which $K_\alpha(u(t_1)) = 0$. Thus, by Lemma 2.4, we get $E(U_0) + cM(U_0) = E(U(t_1)) + cM(U(t_1)) \geq V(u(t_1)) \geq d(c)$ implying that U_0 is not in $\widetilde{\Sigma}_\alpha$, which is a contradiction. □

3. Instability of Traveling Waves

We first compute $d(c)$ and some related quantities. It follows from (2.13) and Lemma 2.3 that

$$\begin{aligned} d(c) &= V(\phi_c) \\ &= \frac{A}{2}\|\phi_c\|_{L^2}^2 + \frac{B}{2}\|\phi_c'\|_{L^2}^2 - \frac{1}{p+1}\|\phi_c\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{3.1}$$

Using the Pohozaev identities, (2.8) and (2.9), in this equation yields

$$d(c) = A\left(\frac{p-1}{p+3}\right)\|\phi_c\|_{L^2}^2 . \tag{3.2}$$

We observe from (2.7) that ϕ_c and ϕ_0 are related through the scaling:

$$\phi_c(x) = A^{\frac{1}{p-1}}\phi_0(a^{\frac{1}{2}}A^{\frac{1}{2}}B^{-\frac{1}{2}}x),$$

so that

$$\|\phi_c\|_{L^2}^2 = a^{-\frac{1}{2}}A^{\frac{5-p}{2p-2}}B^{\frac{1}{2}}\|\phi_0\|_{L^2}^2 .$$

Substituting this into (3.2) we obtain

$$d(c) = a^{-\frac{1}{2}}(1-c^2)^{\frac{p+3}{2(p-1)}}(a-bc^2)^{\frac{1}{2}}d(0), \tag{3.3}$$

where

$$d(0) = \frac{p-1}{p+3}\|\phi_0\|_{L^2}^2 > 0.$$

Our main result is the following theorem showing that traveling waves with $c^2 < c_0^2$ are unstable by blow-up in a finite time.

Theorem 3.1. *Suppose $c^2 < c_0^2$ where c_0^2 is given by (1.14), and ϕ_c is a traveling wave solution of (1.1) with velocity c. Let $\Phi_c = (\phi_c, -c\phi_c)$ be the corresponding solution of (2.1)-(2.2). There exists initial data U_0 arbitrarily close to Φ_c in $H^1\times H^1$ such that the $H^1\times H^1$ norm of the solution $U(t) = (u(t), w(t))$ of (2.1)-(2.3) blows up in finite time.*

Proof. We consider the solution $\Phi_c = (\phi_c, -c\phi_c)$ of (2.1)-(2.2), corresponding to the traveling wave solution ϕ_c. For $\lambda > 1$, we let

$$\begin{aligned} h(\lambda) &= E(\lambda\Phi_c) + cM(\lambda\Phi_c) = V(\lambda\phi_c) \\ &= \frac{1}{2}\left(A\|\phi_c\|_{L^2}^2 + B\|\phi_c'\|_{L^2}^2\right)\lambda^2 - \frac{1}{p+1}\|\phi_c\|_{L^{p+1}}^{p+1}\lambda^{p+1}, \end{aligned}$$

where we have used (2.13) and (2.14). The function $h(\lambda)$ has a local maximum at

$$\lambda_{\max} = \left(\frac{A\|\phi_c\|_{L^2}^2 + B\|\phi_c'\|_{L^2}^2}{\|\phi_c\|_{L^{p+1}}^{p+1}}\right)^{\frac{1}{p-1}} .$$

The Pohozaev identity (2.8) implies that $\lambda_{\max} = 1$. Then, for $\lambda > 1$ (λ near 1) we have

$$E(\lambda\Phi_c) + cM(\lambda\Phi_c) < V(\phi_c) = d(c). \tag{3.4}$$

As $\lambda^{p+1} > \lambda^2$, using (2.15) we get

$$\begin{aligned} K_\alpha(\lambda\phi_c) &= \lambda^2\frac{A}{2}(2\alpha+1)\|\phi_c\|_{L^2}^2 + \lambda^2\frac{B}{2}(2\alpha-1)\|\phi_c'\|_{L^2}^2 - \lambda^{p+1}(\alpha + \frac{1}{p+1})\|\phi_c\|_{L^{p+1}}^{p+1} \\ &< \lambda^2 K_\alpha(\phi_c) = 0. \end{aligned}$$

The above two results imply that $\lambda\Phi_c \in \widetilde{\Sigma}_\alpha$. We now choose a function v_0 such that

$$\widehat{v_0}(\xi) = \begin{cases} \frac{1}{i\xi}\lambda\widehat{\phi_c}(\xi) & \text{for } |\xi| \geq h > 0, \\ 0 & \text{for } |\xi| < h \end{cases}$$

and set $U_0 = ((v_0)_x, -c(v_0)_x)$. We note that $\|U_0 - \Phi_c\|_{H^1\times H^1}$ can be made arbitrarily small by choosing $\lambda - 1$ and h sufficiently small. Thus, by continuity of the functionals, we get that $U_0 \in \widetilde{\Sigma}_\alpha$. By Lemma 2.5 it follows that the solution of (2.1)-(2.3) with initial value U_0 stays in $\widetilde{\Sigma}_\alpha$ as long as it exists: $U(t) = (u(t), w(t)) \in \widetilde{\Sigma}_\alpha$. Also, using (2.5), (2.8), (2.9) and $d(c) = V(\phi_c)$ we obtain

$$\begin{aligned} -2cM(\Phi_c) &= -2cM(\phi_c, -c\phi_c) = 2c^2(\|\phi_c\|_{L^2}^2 + b\,\|\phi_c'\|_{L^2}^2) \\ &= 2c^2\left[1 + \frac{b(p-1)(1-c^2)}{(p+3)(a-bc^2)}\right]\|\phi_c\|_{L^2}^2 \\ &= \frac{2c^2}{1-c^2}\left[1 + \frac{b(p-1)(1-c^2)}{(p+3)(a-bc^2)}\right]\left(\frac{p+3}{p-1}\right)d(c). \end{aligned}$$

Consequently, for $\lambda > 1$, we have

$$-2cM(\lambda\Phi_c) > -2cM(\Phi_c) = \frac{2c^2}{1-c^2}\left[1 + \frac{b(p-1)(1-c^2)}{(p+3)(a-bc^2)}\right]\left(\frac{p+3}{p-1}\right)d(c).$$

Again, by continuity, this leads to the following estimate that will be used later:

$$-2cM(U_0) > \frac{2c^2}{1-c^2}\left[1 + \frac{b(p-1)(1-c^2)}{(p+3)(a-bc^2)}\right]\left(\frac{p+3}{p-1}\right)d(c). \tag{3.5}$$

We now define

$$H(t) = \frac{1}{2}\left(\|v(t)\|_{L^2}^2 + b\,\|u(t)\|_{L^2}^2\right),$$

where v is defined as

$$v(t) = v_0 + \int_0^t w(\tau)d\tau.$$

Note that, due to (2.1), $u = v_x$ and $w = v_t$. We will now show that $H(t)$ blows up in finite time. As in [7], this will ensures that the solution $U(t)$ will blow up in $H^1 \times H^1$ in finite time. To this end we employ Levine's Lemma [16] and start by estimating $H''(t)$. For convenience we suppress the dependencies on t from now on. Since $v_t = w$ and $u_t = w_x$, using (2.2) we get

$$H' = \langle v, w\rangle + b\langle u, w_x\rangle, \tag{3.6}$$

$$H'' = \|w\|_{L^2}^2 + b\,\|w_x\|_{L^2}^2 - \|u\|_{L^2}^2 - a\,\|u_x\|_{L^2}^2 + \|u\|_{L^{p+1}}^{p+1}. \tag{3.7}$$

From the energy conservation we have

$$E(U) = \frac{1}{2}\left(\|w\|_{L^2}^2 + b\,\|w_x\|_{L^2}^2 + \|u\|_{L^2}^2 + a\,\|u_x\|_{L^2}^2\right) - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} = E(U_0).$$

Eliminating $\|u\|_{L^{p+1}}^{p+1}$ in (3.7) we get

$$\begin{aligned} H''(t) &= \frac{p+3}{2}\left(\|w + cu\|_{L^2}^2 + b\|w_x + cu_x\|_{L^2}^2\right) - 2cM(U_0) \\ &\quad -(p+1)[E(U_0) + cM(U_0)] + J_c(u), \end{aligned} \tag{3.8}$$

where

$$J_c(u) = \frac{p-1}{2}\left\{\left[1 - c^2\left(\frac{p+3}{p-1}\right)\right]\|u\|_{L^2}^2 + \left[a - bc^2\left(\frac{p+3}{p-1}\right)\right]\|u_x\|_{L^2}^2\right\}. \tag{3.9}$$

To control $J_c(u)$ we first claim that there are constants $\alpha > \frac{1}{2}$ and $C > 0$ such that

$$J_c(u) = C\left[V(u) - \frac{1}{\alpha(p+1)+1}K_\alpha(u)\right].$$

Note that the coefficient of $K_\alpha(u)$ is chosen so that the term $\|u\|_{L^{p+1}}^{p+1}$ disappears. We then have

$$\begin{aligned}
&V(u) - \frac{1}{\alpha(p+1)+1}K_\alpha(u) \\
&\quad = \frac{(1-c^2)}{2}\left(\frac{\alpha(p-1)}{\alpha(p+1)+1}\right)\|u\|_{L^2}^2 + \frac{a-bc^2}{2}\left(\frac{\alpha(p-1)+2}{\alpha(p+1)+1}\right)\|u_x\|_{L^2}^2 \\
&\quad = \frac{1}{C}\left\{\frac{p-1}{2}\left[1-c^2\left(\frac{p+3}{p-1}\right)\right]\|u\|_{L^2}^2\right. \\
&\qquad \left. + \frac{(a-bc^2)}{2(1-c^2)\alpha}\left[1-c^2\left(\frac{p+3}{p-1}\right)\right][\alpha(p-1)+2]\|u_x\|_{L^2}^2\right\} \qquad (3.10)
\end{aligned}$$

where we set

$$C = \frac{[\alpha(p+1)+1]}{(1-c^2)\alpha}\left[1-c^2\left(\frac{p+3}{p-1}\right)\right]. \qquad (3.11)$$

The coefficient of $\|u_x\|_{L^2}^2$ inside curly brackets in (3.10) is the same with that of (3.9), if we choose α as follows:

$$\alpha = \frac{(a-bc^2)}{2c^2(a-b)}\left[1-c^2\left(\frac{p+3}{p-1}\right)\right]. \qquad (3.12)$$

Hence, combining (3.11) and (3.12) gives

$$C = \frac{(a-bc^2)\left[1-c^2\left(\frac{p+3}{p-1}\right)\right](p+1)+2c^2(a-b)}{(1-c^2)(a-bc^2)}. \qquad (3.13)$$

To ensure $\alpha > \frac{1}{2}$ we must have

$$(a-bc^2)\left[1-c^2\left(\frac{p+3}{p-1}\right)\right] > c^2(a-b).$$

This can be simplified as follows

$$k(c^2) = b(p+3)c^4 - 2a(p+1)c^2 + a(p-1) > 0. \qquad (3.14)$$

Since $k(0) > 0$ and $k(1) < 0$, the function $k(c^2)$ has only one zero on the interval $(0,1)$. Then, (3.14) is satisfied if $c^2 < c_0^2$ with

$$\begin{aligned}
c_0^2 &= \frac{a}{b}\left(\frac{p+1}{p+3}\right)\left[1-\left(1-\frac{b(p+3)(p-1)}{a(p+1)^2}\right)^{1/2}\right] \\
&= \left(\frac{p-1}{p+1}\right)\left[1+\left(1-\frac{b(p+3)(p-1)}{a(p+1)^2}\right)^{1/2}\right]^{-1}.
\end{aligned}$$

Finally, it follows from (3.13) that $C > 0$ since

$$c_0^2 \le \frac{p-1}{p+3}.$$

We next claim that $J_c(u) \ge Cd(c)$. Since $u \in \widetilde{\Sigma}_\alpha$, we have $K_\alpha(u) < 0$. We can then find $0 < \gamma < 1$ so that $K_\alpha(\gamma u) = 0$. By Lemma 2.3, this implies that $V(\gamma u) \ge d(c)$. But then

$$\begin{aligned}
J_c(u) &> \gamma^2 J_c(u) = J_c(\gamma u) = C\left[V(\gamma u) - \frac{1}{\alpha(p+1)+1}K_\alpha(\gamma u)\right] \\
&= CV(\gamma u) \ge Cd(c) \qquad (3.15)
\end{aligned}$$

which proves our claim. We are now in the position of putting all the above calculations together to estimate H''. Writing $E(U_0)+cM(U_0)=d(c)-\delta$ with $\delta>0$ in (3.8) and using (3.5), (3.15), we get

$$\begin{aligned} H'' &\geq \frac{p+3}{2}\left(\|w+cu\|_{L^2}^2+b\|w_x+cu_x\|_{L^2}^2\right)-(p+1)d(c)+(p+1)\delta \\ &\quad +\frac{2c^2}{1-c^2}\left[1+\frac{b(p-1)(1-c^2)}{(p+3)(a-bc^2)}\right]\left(\frac{p+3}{p-1}\right)d(c)+Cd(c) \\ &= \frac{p+3}{2}\left(\|w+cu\|_{L^2}^2+b\|w_x+cu_x\|_{L^2}^2\right)+(p+1)\delta+\sigma d(c) \end{aligned}$$

where

$$\begin{aligned} \sigma &= -(p+1)+\frac{2c^2}{1-c^2}\left[1+\frac{b(p-1)(1-c^2)}{(p+3)(a-bc^2)}\right]\left(\frac{p+3}{p-1}\right) \\ &\quad +\frac{(a-bc^2)\left[1-c^2\left(\frac{p+3}{p-1}\right)\right](p+1)+2c^2(a-b)}{(1-c^2)(a-bc^2)}. \end{aligned}$$

A direct calculation shows that σ is zero to yield

$$H''\geq\frac{p+3}{2}\left(\|w+cu\|_{L^2}^2+b\|w_x+cu_x\|_{L^2}^2\right)+(p+1)\delta.$$

So, $H''(t)>(p+1)\,\delta$ which in turn implies that $H'(t_0)>0$ for some $t_0>0$. Since $u=v_x$ we have $\langle v,u\rangle=\langle u,u_x\rangle=0$. Then from (3.6)

$$H'=\langle v,w+cu\rangle+b\langle u,w_x+cu_x\rangle.$$

Thus

$$(H')^2\leq\left(\|v\|_{L^2}^2+b\|u\|_{L^2}^2\right)\left(\|w+cu\|_{L^2}^2+b\|w_x+cu_x\|_{L^2}^2\right).$$

Finally, we have

$$HH''-\frac{p+3}{4}\left(H'\right)^2\geq(p+1)H\delta\geq 0.$$

By Levine's Lemma [16] this shows that $H(t)$ blows up in finite time. This completes the proof. $\square$

4. Stability Regions for Traveling Waves

In this section we investigate both analytically and numerically the dependence of stability regions of traveling wave solutions of the double dispersion equation on the parameters a, b and p. To be precise, by the stability region we mean the set of wave velocities c for which the traveling wave solutions of (1.1) are orbitally stable. Recall that a traveling wave ϕ_c is said to be orbitally stable if any solution $U(t)$ with initial data sufficiently close to the traveling wave stays close, at any later time, to some translate of ϕ_c. It is a well-known phenomena in nonlinear wave theory that orbital stability occurs for all values of c for which a scalar function $d(c)$ is convex [17, 18, 19]. For the general class given by (1.2), this was proved explicitly in [8]. To apply the convexity criterion to the double dispersion equation, we first rewrite the function $d(c)$ given in (3.3) as

$$d(c)=d(0)(1-c^2)^{\frac{p+3}{2(p-1)}}(1-\mu c^2)^{\frac{1}{2}},\ \ \mu=\frac{b}{a}. \tag{4.1}$$

As $0 \le b < a$, we will consider $0 \le \mu < 1$. A direct computation of $d''(c)$ gives

$$d''(c) = d(0)(p-1)^{-2}(1-c^2)^{\frac{7-3p}{2(p-1)}}(1-\mu c^2)^{-3/2}(Pc^6 - Qc^4 + Rc^2 - S) \tag{4.2}$$

with

$$\begin{aligned} P &= 2(p+3)(p+1)\mu^2, \\ Q &= 3(p+3)(p-1)\mu^2 + (3p^2+10p+19)\mu, \\ R &= 2((3p+5)(p-1)\mu + 2(p+3)), \\ S &= (p-1)^2\mu + (p-1)(p+3). \end{aligned}$$

Hence the sign of $d''(c)$ is determined by the sign of the polynomial

$$G(z,p,\mu) = Pz^3 - Qz^2 + Rz - S. \tag{4.3}$$

Recalling that traveling waves exist for $c^2 < 1$, we see that the stability regions are the set of all wave velocities c for which $c^2 < 1$ and $G(c^2,p,\mu) > 0$. So the problem reduces to the problem of finding real roots of $G(z,p,\mu)$ on the interval $(0,1)$. The remainder of this section focuses on analyzing how the parameters p and μ affect the locations of the roots and, consequently, the stability regions. We first restrict our attention to the exploration of locations and number of the roots in $(0,1)$ and then focus on formulating explicit stability conditions in terms of c in the last part of this section.

First we observe that the coefficients P, Q, R and S are all positive, so all real roots of $G(z,p,\mu)$ must be positive and $G(0,p,\mu) < 0$. As P, Q, R, S are continuous in the parameters p and μ, the three (possibly complex) roots $z(p,\mu)$ of the cubic polynomial $G(z,p,\mu)$ depend continuously on p and μ for $p > 1$ and $\mu > 0$.

It will be useful to consider what happens at $z = 1$. Computation gives

$$G(1,p,\mu) = (\mu-1)^2(p+3)(5-p). \tag{4.4}$$

Hence, for $\mu < 1$, $G(1,p,\mu) > 0$ when $p < 5$ but $G(1,p,\mu) < 0$ when $p > 5$. Since $G(0,p,\mu) < 0$, the number of distinct roots of $G(z,p,\mu)$ in the interval $(0,1)$ must be *(i)* one or three when $p < 5$ and *(ii)* zero or two when $p > 5$.

Equation (4.4) shows that when $p = 5$ we have the root $z_1(5,\mu) = 1$. This will allow us to determine completely the case $p = 5$. Factoring $G(z,5,\mu)$, we get

$$G(z,5,\mu) = 16(z-1)(6\mu^2z^2 - 9\mu z + \mu + 2), \tag{4.5}$$

which yields the other two distinct roots

$$z_\pm(5,\mu) = \frac{1}{12\mu}(9 \pm \sqrt{33 - 24\mu}). \tag{4.6}$$

We now try to locate the roots $z_\pm(5,\mu)$ of $G(z,5,\mu)$ in $(0,1)$. Since $0 \le \mu < 1$, the roots $z_\pm(5,\mu)$ are real and, in consequence, there are three real roots. First note that $\mu < 1$ implies $z_+(5,\mu) \ge \frac{1}{\mu} > 1$. On the other hand, an easy computation shows that $z_-(5,\mu) < 1$ if and only if $\frac{1}{3} < \mu < 1$. Summing up, we have: $G(z,5,\mu)$ has one root $z_-(5,\mu)$ in $(0,1)$ for $\frac{1}{3} < \mu < 1$ but no root in $(0,1)$ for $0 \le \mu \le \frac{1}{3}$.

Next we want to use continuity of the roots with respect to the parameters to understand what happens when p is near 5 and μ is fixed. We first decrease p slightly from 5. Recall that $G(z,p,\mu)$ must have exactly one root or three roots in $(0,1)$ for $p < 5$. Since $z_+(5,\mu) > 1$ for $\mu < 1$, we cannot have the case of three roots. Therefore, for p slightly smaller than 5 and $0 \le \mu < 1$, $G(z,p,\mu)$ will have exactly one root in $(0,1)$. To determine what happens when p increases slightly from 5, we have to consider two cases: $\mu < \frac{1}{3}$ and $\mu > \frac{1}{3}$. When $\mu < \frac{1}{3}$, $G(z, 5, \mu)$ has two roots $z_-(5, \mu)$ and $z_+(5, \mu)$ in $(1, \infty)$. For p sufficiently close to 5, none of these two roots can move into $(0, 1)$. Recalling that for $p > 5$, $G(z, p, \mu)$ must have zero or two roots in $(0, 1)$, and noting that we have just eliminated the possibility of two roots when p is slightly greater than 5 and $\mu < \frac{1}{3}$, we conclude that $G(z, p, \mu)$ has no root in $(0, 1)$. Finally, consider the case $\mu > \frac{1}{3}$ with p slightly larger than 5. Since $z_-(5, \mu) < z_1(5, \mu) = 1$, the only possibility is that the root $z_1(5, \mu)$ moves to the left, yielding exactly two roots in $(0, 1)$ for $G(z, p, \mu)$.

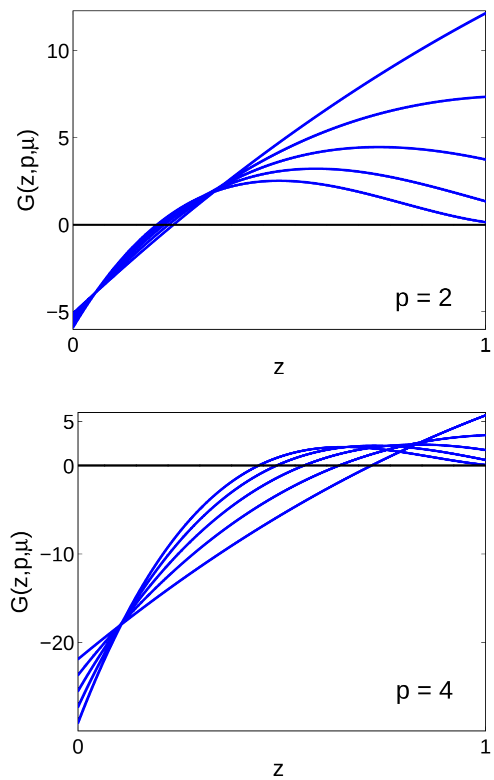
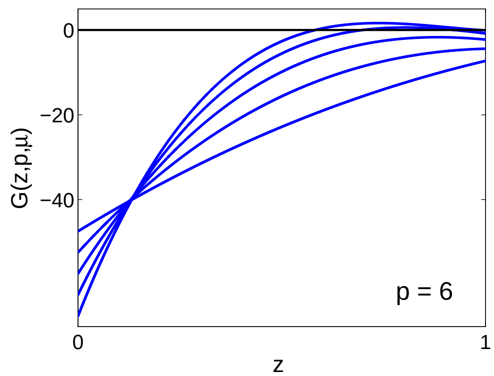


FIGURE 1. Variation of the function $G(z, p, \mu)$ with z on the interval $[0, 1]$ for (a) $p = 2$, (b) $p = 4$ and for $\mu = 0.1, 0.3, 0.5, 0.7, 0.9$ (from top to bottom at the right end-point).

Summing up what we know about the total number of roots on the interval $(0, 1)$, we have:

- For any $0 \le \mu < 1$, $G(z, p, \mu)$ has only one root in $(0, 1)$ when $p < 5$ and p near 5.
- For $\mu < \frac{1}{3}$, $G(z, p, \mu)$ has no root in $(0, 1)$ when $p > 5$ and p near 5.
- For $\mu > \frac{1}{3}$, $G(z, p, \mu)$ has two roots in $(0, 1)$ when $p > 5$ and p near 5.

For general values of p and μ we now provide numerical evidence to suggest that exactly the same behavior is observed for other parameter values. In Figures 1 and 2 we present the graph of $G(z, p, \mu)$ as a function of z on $[0, 1]$ for $p = 2, 4$ and



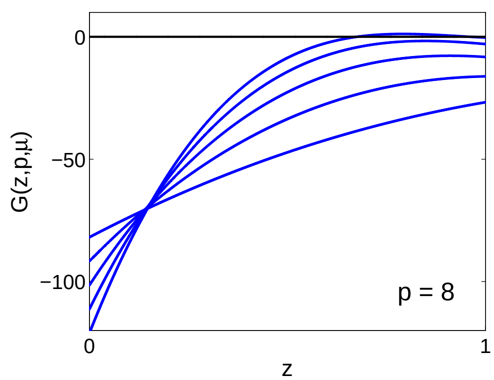


FIGURE 2. Variation of the function $G(z,p,\mu)$ with z on the interval $[0,1]$ for (a) $p=6$, (b) $p=8$ and for $\mu = 0.1, 0.3, 0.5, 0.7, 0.9$ (from bottom to top at the right end-point).

$p = 6, 8$, respectively. In each figure, the curves correspond to the following five cases: $\mu = 0.1, 0.3, 0.5, 0.7, 0.9$. The curves are identified from (4.4) by observing that $G(1,p,\mu)$ is decreasing in μ for $p < 5$ but increasing in μ for $p > 5$. That is, at the right end-point, the curves correspond to $\mu = 0.1, 0.3, 0.5, 0.7, 0.9$ from top to bottom for $p < 5$ but from bottom to top for $p > 5$, respectively. We see from the figures that the itemized conclusions of the previous paragraph about the number of roots of $G(z,p,\mu)$ for p near 5 are exactly valid for all values of p and μ with a critical value μ_p replacing the value $1/3$. Motivated by this fact, we will make the following claim about the number of roots of $G(z,p,\mu)$ in $(0,1)$:

- For $p < 5$ and $0 \le \mu < 1$, $G(z,p,\mu)$ has only one root $z_1(p,\mu)$ in $(0,1)$.
- For $p > 5$ there is a critical value $\mu_p \in (0,1)$ so that $G(z,p,\mu)$ has no root in $(0,1)$ for $0 \le \mu < \mu_p$ but it has two roots $z_1(p,\mu), z_2(p,\mu)$ in $(0,1)$ for $\mu_p < \mu < 1$.

We note that the above claim contains the case of the Boussinesq equation ($\mu = 0$), where $G(z,p,0)$ has exactly one root $z_1(p,0) = \frac{p-1}{4}$. This root is in $(0,1)$ if and only if $p < 5$. Another limiting case where $\mu = 1$ was analysed in [8]. In this case we have

$$G(z,p,1) = 2(p+1)(p+3)\left(z - \frac{p-1}{p+3}\right)(z-1)^2, \tag{4.7}$$

which implies that $G(z,p,1)$ has only one root $z = \frac{p-1}{p+3}$ in $(0,1)$. Note that this can be considered as a limiting case of the claim above with $z_2(p,1) = 1$.

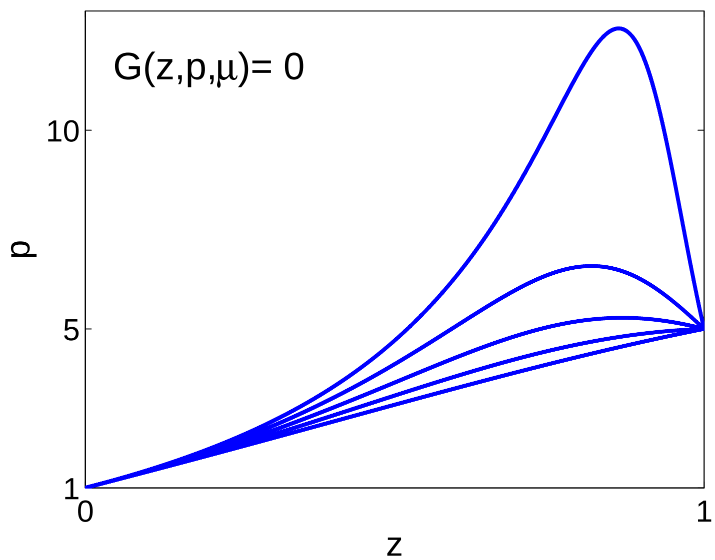


FIGURE 3. Variation of p with z on the interval $[0,1]$ when $G(z,p,\mu) = 0$ for $\mu = 0.1, 0.3, 0.5, 0.7, 0.9$ (from bottom to top).

The information collected for the locations of the roots of $G(z,p,\mu)$ allows us to determine the stability regions as follows:

- If $G(z,p,\mu)$ has no root in $(0,1)$, then the stability region is empty.
- If $G(z,p,\mu)$ has one root $z_1(p,\mu)$ in $(0,1)$, then the stability region is the set of wave velocities satisfying $z_1(p,\mu) < c^2 < 1$.
- If $G(z,p,\mu)$ has two roots $z_1(p,\mu) < z_2(p,\mu)$ in $(0,1)$, then the stability region is the set of wave velocities satisfying $z_1(p,\mu) < c^2 < z_2(p,\mu)$.

To illustrate the roots of $G(z,p,\mu)$ and the corresponding stability regions, we have also graphed the set $G(z,p,\mu) = 0$ in the $zp-$plane for certain fixed values of μ in Fig. 3. The curves are ordered from bottom to top: the bottom one is the set $G(z,p,0) = 0$, and the curves move up as μ increases. The curves show the location of the real roots of $G(z,p,\mu)$ for the corresponding values of μ. Namely, a point (z^*,p^*) on the curve corresponds to the root z^* of $G(z,p^*,\mu)$. Conforming with our conjecture about the roots, the graph indicates that: *(i)* when $p < 5$ there is exactly one root $z_1(p,\mu)$ in $(0,1)$ which decreases as μ increases; *(ii)* when $p > 5$, there is some μ_p such that for $\mu < \mu_p$ there is no root in $(0,1)$ whereas for $\mu > \mu_p$ there are two roots $z_1(p,\mu) < z_2(p,\mu)$ in $(0,1)$. Moreover, $z_1(p,\mu)$ is decreasing in μ, while $z_2(p,\mu)$ increases and approaches 1 as μ approaches 1. For a fixed μ_0, the orbital stability interval is obtained by intersecting the line $p = p_0$

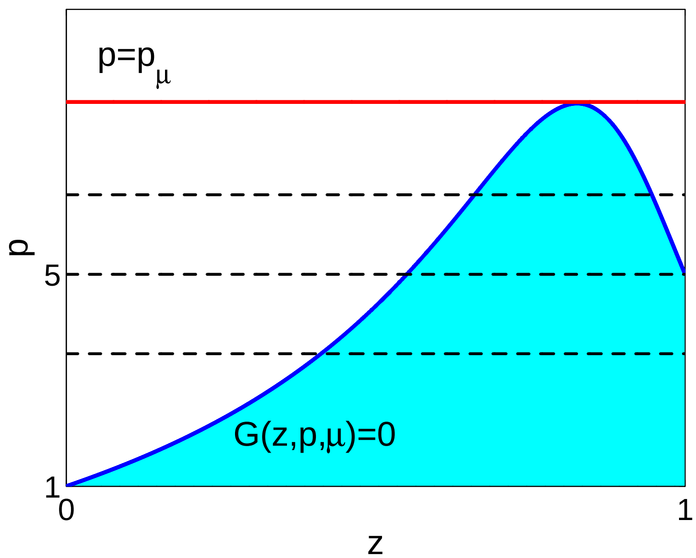


FIGURE 4. Schematic diagram of the stability region (shaded region) for a fixed μ.

with the set $G(z,p,\mu_0) = 0$, this set in turn is either empty or an interval for c^2 of the form $(z_1(p_0,\mu_0),1)$ or $(z_1(p_0,\mu_0),z_2(p_0,\mu_0))$. Fig. 3 also indicates that the critical value μ_p increases with p. This means that we can as well fix μ and vary p. Then there is a critical value $p = p_\mu$ so that when $p \geq p_\mu$ the stability regions are empty. To illustrate this, in Fig. 4 we take a single curve $G(z,p,\mu) = 0$ with fixed μ and several horizontal lines corresponding to different values of p. We observe transitions between different types of stability regions as p varies for a fixed μ. Fig. 4 also gives the critical value p_μ. The shaded region in Fig. 4, that is, the area between the curve $G(z,p,\mu) = 0$ and the line $p = 1$, corresponds to the stability regions of the problem for varying p.

To conclude, our analysis in this section leads to the following observation: *Traveling wave solutions of the double dispersion equation (1.1) are orbitally stable in each of the following three cases;*

$$(A)\ p < 5 \ \textit{and}\ z_1(p,\mu) < c^2 < 1, \tag{4.8}$$

$$(B)\ p = 5,\ \frac{1}{3} < \mu < 1 \ \textit{and}\ \frac{1}{12\mu}(9 - \sqrt{33 - 24\mu}) < c^2 < 1, \tag{4.9}$$

$$(C)\ p > 5,\ \mu_p < \mu < 1 \ \textit{and}\ z_1(p,\mu) < c^2 < z_2(p,\mu) < 1. \tag{4.10}$$

Moreover, for a fixed p, as μ increases, the stability interval gets larger. Also, for $p > 5$, the critical value μ_p increases as p increases.

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Department of Natural and Mathematical Sciences, Faculty of Engineering, Ozyegin University, Cekmekoy 34794, Istanbul, Turkey
E-mail address: husnuata.erbay@ozyegin.edu.tr

Department of Natural and Mathematical Sciences, Faculty of Engineering, Ozyegin University, Cekmekoy 34794, Istanbul, Turkey
E-mail address: saadet.erbay@ozyegin.edu.tr

Faculty of Engineering and Natural Sciences, Sabanci University, Tuzla 34956, Istanbul, Turkey
E-mail address: albert@sabanciuniv.edu