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Stability of solitary waves for three-coupled long wave–short wave interaction equations

HANDAN BORLUK* AND SAADET ERBAY
Department of Mathematics, Isik University, 34980 Sile-Istanbul, Turkey
*Corresponding author: hborluk@isikun.edu.tr

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In this paper, we consider a three-component system of 1D long wave–short wave interaction equations. The system has two-parameter family of solitary wave solutions. We prove orbital stability of the solitary wave solutions using variational methods.

Keywords: long wave–short wave interaction equations; orbital stability of solitary waves.

1. Introduction

Wave propagation problems in various continuous media, such as fluids, solids and optical fibers, lead to single or coupled partial differential equations. While the Korteweg-de Vries-type equations describe propagation of long waves, the non-linear Schrödinger (NLS)-type equations describe propagation of envelope of short waves in continuous media, where a different length scale is defined in each case. If the phase speed of a long wave coincides with the group speed of a short wave, then the resonant interaction between long and short waves occur. This interaction is modelled by a two-component coupled evolution equations called the long wave–short wave interaction (LSI) equations,

$$\left.\begin{aligned} &i\phi_t + \phi_{xx} = \beta w \phi, \\ &w_t = \nu(|\phi|^2)_x, \end{aligned}\right\} \tag{1.1}$$

where $\phi\colon \mathbb{R}\times\mathbb{R}^+ \to \mathbb{C}$, $w\colon \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$, β and ν are real constants. Here, w represents a long-wave mode and ϕ denotes short-wave mode propagating in a continuous medium. System (1.1) was derived to describe the resonant interaction of waves propagating on the surface of water (Djordjevic & Redekopp, 1977). The same system was also obtained for the resonant interaction of internal gravity waves (Grimshaw, 1977). Motivated by the physical significance, various aspects of (1.1), such as solitary wave solutions, stability of solitary wave solutions, well posedness of the Cauchy problem, have been widely considered (Ma, 1978; Tsutsumi & Hatano, 1994; Laurençot, 1995; Ginibre & Tsutsumi, 1997).

On the other hand, if there exist two short waves with the same group speed and a long wave whose phase speed is equal to the group speed of short waves, then long wave–short wave resonant interaction arises among the wave modes. This phenomena is modelled by the three coupled LSI equations

$$\left.\begin{aligned} &i\phi_t + \phi_{xx} = \beta w \phi, \\ &i\psi_t + \psi_{xx} = \beta w \psi, \\ &w_t = \beta(|\phi|^2 + |\psi|^2)_x, \end{aligned}\right\} \tag{1.2}$$

where $\phi, \psi\colon \mathbb{R}\times\mathbb{R}^+ \to \mathbb{C}$, $w\colon \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ and β is a real constant. Here, w represents a long-wave mode and ϕ and ψ denote short-wave modes. System (1.2) is a generalization of the two component LSI system and appears, for instance, in water waves (Craik, 1985) and in a bulk elastic medium (Erbay, 2000). Motivated by physical applications, various aspects of (1.2) are investigated analytically and numerically (Ma, 1981; Eden *et al.*, 2005; Borluk *et al.*, 2007). The main purpose of the present study is to show that solitary wave solutions of (1.2) are stable in some sense.

The LSI system (1.2) has a two-parameter family of solitary wave solutions of the form

$$\left.\begin{aligned} \phi_s(x,t) &= \Phi(x-ct)\mathrm{e}^{i\omega t},\\ \psi_s(x,t) &= \Psi(x-ct)\mathrm{e}^{i\omega t},\\ w_s(x,t) &= W(x-ct), \end{aligned}\right\} \tag{1.3}$$

where $W(x) = -\beta(|\Phi(x)|^2+|\Psi(x)|^2)/c$, $(\Phi(x), \Psi(x)) = (R_1(x), R_2(x))\mathrm{e}^{\frac{icx}{2}}$, $c > 0$ and $4\omega - c^2 > 0$. Here, $W \in L^2(\mathbb{R})$ and $(R_1, R_2) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ are positive solutions of

$$\left.\begin{aligned} -u_{xx} + \left(\omega - \frac{c^2}{4}\right)u - \frac{\beta^2}{c}(u^2+v^2)u = 0,\\ -v_{xx} + \left(\omega - \frac{c^2}{4}\right)v - \frac{\beta^2}{c}(u^2+v^2)v = 0. \end{aligned}\right\} \tag{1.4}$$

The mathematically exact theory for stability of solitary waves dates back to (Benjamin, 1972) for the Korteweg-de Vries-type equations. In that work, a Lyapunov functional was constructed using the conserved quantities of the Korteweg-de Vries equation, and it was shown that the stability of solitary waves relied on suitable lower and upper bounds on the variation of the Lyapunov functional. In a later study (Weinstein, 1986), the same method has been used to show the stability of standing waves of the NLS equation, which has been already proved in Cazenave & Lions (1982) using the concentration–compactness methods. In Laurençot (1995), using the so-called Lyapunov method, stability of solitary wave solutions of (1.1), $\phi(x,t) = R(x-ct)\mathrm{e}^{i\omega t + i\frac{c}{2}(x-ct)}$, $w(x,t) = W(x-ct)$, where $(R, W) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, was established when $c > 0$ and $4\omega - c^2 > 0$. In the present paper, our aim is to extend the above method to the three component LSI system, and show that the solitary waves of (1.2) are orbitally stable.

The organization of the paper is as follows: The local well posedness of the Cauchy problem for (1.2) is discussed, and conserved integrals for the same system is given in Section 2. A variational characterization of the solitary waves, which will be used in the proof of the stability of solitary wave solutions, is briefly presented in Section 3. We state the stability theorem that relies on a lower bound of the second variation of the Lyapunov functional in Section 4. Using the analysis of the unconstrained variational problem presented briefly in Section 3, the lower bound is proved and the stability of solitary waves is established in the same section.

Notations. Throughout the paper $L^p(\mathbb{R})$, $1 \leqslant p < \infty$, represents the space of p-integrable functions. $\|f\|_p$ denotes the $L^p(\mathbb{R})$ norm of f, $1 \leqslant p \leqslant \infty$. $H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$ is the Sobolev space of f for which the norm $\|f\|_{H^1}^2 = \|f\|_2^2 + \|\nabla f\|_2^2$ is finite. $\langle f, g\rangle$ refers to the inner product of f and g in $L^2(\mathbb{R})$.

2. Local well posedness of Cauchy problem

The Cauchy problem for the two component LSI system (1.1) was studied in Tsutsumi & Hatano (1994) for initial data $(\phi_0, w_0) \in H^{1/2}(\mathbb{R}) \times L^2(\mathbb{R})$. A contraction technique together with smoothing effect estimates (Kenig *et al.*, 1991, 1993) was used to prove existence and uniqueness of solutions of the initial value problem in suitable Banach spaces. Introducing a regularized system, the existence and and uniqueness results for (1.1) was also established in Laurençot (1995) for the initial data $(\phi_0, w_0) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ that was essential to the study of orbital stability of solitary waves. Later, the local-well-posedness result for (1.1) was improved in Ginibre & Tsutsumi (1997) for initial data $(\phi_0, w_0) \in H^k(\mathbb{R}) \times L^{1/k}(\mathbb{R})$, $0 < k < 1/2$.

The Cauchy problem for the three component LSI system (1.2)

$$\left.\begin{array}{l} i\phi_t + \phi_{xx} = F_\phi(\phi, \psi), \\ i\psi_t + \psi_{xx} = F_\psi(\phi, \psi), \\ \phi(x,0) = \phi_0(x), \quad \psi(x,0) = \psi_0(x) \end{array}\right\} \tag{2.1}$$

was considered in Eden *et al.* (2005) for the initial data $(\phi_0, \psi_0, w_0) \in H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \times L^2(\mathbb{R})$, where

$$(F_\phi, F_\psi) = \left(\beta^2 \int_0^t (|\phi(x,s)|^2 + |\psi(x,s)|^2)_x \, \mathrm{d}s + \beta w_0(x)\right)(\phi(x,t), \psi(x,t)).$$

In that study, following Tsutsumi & Hatano (1994), a fixed point method was used to establish the existence and uniqueness of local in time solutions of (1.2) in a suitable Banach space:

THEOREM 1 Let $(\phi_0, \psi_0) \in H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ and $w_0 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. There exists a unique solution $(\phi(x,t), \psi(x,t))$ of the Cauchy problem (2.1) on $[0, T]$ for $T > 0$ such that $\phi \in C\left([0,T]; H^{1/2}(\mathbb{R})\right)$, $\phi_x \in L^\infty\left(\mathbb{R}; L^2[0,T]\right)$, $\psi \in C\left([0,T]; H^{1/2}(\mathbb{R})\right)$ and $\psi_x \in L^\infty\left(\mathbb{R}; L^2[0,T]\right)$.

The conserved integrals of the LSI system (1.2) are of the form (Borluk *et al.*, 2007)

$$\begin{aligned} I_1 &= \int_{\mathbb{R}} |\phi|^2 \, \mathrm{d}x, \quad I_2 = \int_{\mathbb{R}} |\psi|^2 \, \mathrm{d}x, \\ I_3 &= \int_{\mathbb{R}} [w^2 + i(\phi^*\phi_x - \phi\phi_x^* + \psi^*\psi_x - \psi\psi_x^*)] \, \mathrm{d}x, \\ I_4 &= \int_{\mathbb{R}} [|\phi_x|^2 + |\psi_x|^2 + \beta(|\phi|^2 + |\psi|^2)w] \mathrm{d}x, \end{aligned} \tag{2.2}$$

where I_1 and I_2 are the mass functionals, I_3 is the momentum functional and I_4 is the energy functional, i.e. the Hamiltonian. It should be pointed out that the conservation of mass, I_1 and I_2, and momentum, I_3, make sense since $(\phi(x,t), \psi(x,t), w(x,t)) \in H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \times L^2(\mathbb{R})$, whereas the the conservation of energy does not. Because the energy functional, I_4, plays a major role in the orbital stability computations, and the natural space for energy is $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R})$, we will assume in the rest of the present study that if the initial data (ϕ_0, ψ_0, w_0) are chosen from $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R})$, then the corresponding solution $(\phi(x,t), \psi(x,t), w(x,t))$ remains in the same space.

3. Variational characterization of solitary waves

In this section, we briefly discuss a variational characterization of solutions for (1.4), which plays a key role in the stability analysis of solitary waves (1.3).

Motivated by Nagy inequality (Nagy, 1941) given as

$$\left(\frac{s}{2}H\left(\frac{s}{\beta},\frac{p-1}{p}\right)\right)^{-\frac{\beta}{s}} \leqslant \frac{\|u_x\|_p^{\frac{\beta}{s}}\|u\|_q^{q+\beta\frac{q(p-1)}{ps}}}{\|u\|_{q+\beta}^{q+\beta}}, \quad u \in H^1(\mathbb{R}), \tag{3.1}$$

where $q, \beta > 0$, $p \geqslant 1$, $s = 1+q(p-1)/p$, $H(a,b) = [(a+b)^{-(a+b)}\Gamma(1+a+b)]/[a^{-a}b^{-b}\Gamma(1+a)\Gamma(1+b)]$ and Γ is the Gamma function and by Gagliardo–Nirenberg inequality

$$\|u\|_r \leqslant C\|\nabla u\|_2^{\vartheta}\|u\|_2^{1-\vartheta}, \quad 0 < \vartheta \leqslant 1, \quad u \in H^1(\mathbb{R}^n),$$

where $\vartheta = n(1/2 - 1/r)$; the non-linear functional $J(u,v)$ on $H^1(\mathbb{R}) \times H^1(\mathbb{R})$

$$J(u,v) = \frac{(\|u\|_2^2 + \|v\|_2^2)^{1-\theta/2}(\|u_x\|_2^2 + \|v_x\|_2^2)^{\theta/2}}{\|u^2+v^2\|_2^{1/2}}, \quad \theta = \frac{1}{4}, \tag{3.2}$$

is defined. The functional $J(u,v)$ is well defined on $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ due to embedding of $H^1(\mathbb{R})$ in $L^4(\mathbb{R})$. It should be pointed out that the non-linear functional $J(u,v)$ is a generalization of the single variable functional $J(u)$ that was considered in the study of standing waves of the NLS equation (Weinstein, 1983).

The first variation of the non-linear functional $J(u,v)$ is given as

$$\delta J = -B\int_{\mathbb{R}}\{[u_{xx} - \Omega u + \gamma(u^2+v^2)u]\eta_1 + [v_{xx} - \Omega v + \gamma(u^2+v^2)v]\eta_2\}\mathrm{d}x,$$

where $\eta_i \in C_0^\infty(\mathbb{R})(i=1,2)$, $\Omega = \omega - c^2/4$ and $\gamma = \beta^2/c$, $B = \left[3^3/\left(4^4\Omega^3\gamma^4\left(\int_{\mathbb{R}}(u^2+v^2)^2\,\mathrm{d}x\right)^6\right)\right]^{1/8}$, and the Pohozaev type identities,

$$3\int_{\mathbb{R}}(u_x^2+v_x^2)\mathrm{d}x = \Omega\int_{\mathbb{R}}(u^2+v^2)\mathrm{d}x = \frac{3\gamma}{4}\int_{\mathbb{R}}(u^2+v^2)^2\mathrm{d}x, \tag{3.3}$$

satisfied by (u,v) are used. It can be shown that the infimum of $J(u,v)$ is achieved at a pair of positive functions (R_1, R_2) when $c > 0$ and $4\omega - c^2 > 0$ using Lieb's compactness lemma (Lieb, 1983). Thus, the critical points of the functional $J(u,v)$ in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ are the non-trivial weak solutions of (1.4). Details of the proof will be given elsewhere.

It should be noted that there are various studies in the literature devoted to the problem of existence of solutions of the coupled system (1.4) and its generalizations (Maia *et al.*, 2006; Figueiredo & Lopes, 2008, and the references therein). In those studies, variational approaches based on minimization of energy functionals subject to some constraints are used. Though the approach presented above is different from those of Maia *et al.* (2006) and Figueiredo & Lopes (2008), it is readily seen that minimizing the energy functional is equivalent to minimizing the non-linear functional $J(u,v)$. Indeed, the energy functional for solitary waves

$$I_4(u,v) = \int_{\mathbb{R}}\left(u_x^2 + v_x^2 + \frac{c^2}{4}(u^2+v^2) - \gamma(u^2+v^2)^2\right)\mathrm{d}x,$$

after the scale transformation $(u_q(x), v_q(x)) = \sqrt{q}(u(qx), v(qx))$ with $q > 0$, takes the form

$$\begin{aligned} I_4(u,v) &\geqslant \inf_{q>0} I_4(u_q, v_q) = \inf_{q>0} \int_{\mathbb{R}} [q^2(u_x^2 + v_x^2) + \frac{c^2}{4}(u^2+v^2) - \gamma q (u^2+v^2)^2] \mathrm{d}x \\ &\geqslant \int_{\mathbb{R}} (q^2(u_x^2+v_x^2) - \gamma q (u^2+v^2)^2) \mathrm{d}x, \end{aligned} \tag{3.4}$$

where the conserved mass integrals do not change, $\|u_q\|_2 = \|u\|_2$ and $\|v_q\|_2 = \|v\|_2$. Using the scaled forms of the identities (3.3) in (3.4), the energy functional takes the form

$$I_4(u,v) \geqslant \inf_{q>0} I_4(u_q, v_q) \geqslant -\left(\frac{3\gamma^2 \Omega^7}{16}\right)^{1/8} \lambda^{5/4} \frac{1}{\inf J(u,v)},$$

for which $J(u_q, v_q) = J(u,v)$ and $\lambda = I_1 + I_2$. Thus, ground state solutions (u_q, v_q), i.e. a minimizer of the Hamiltonian I_4, is also a minimizer of the functional $J(u,v)$.

4. Stability of solitary waves

In this section, we are concerned with the stability of solitary wave solutions (1.3) of system (1.2). For solitary waves, the appropriate notion of stability is orbital stability. All solitary waves of the same form but in different positions through space translation and phase rotation are assumed to be in the same orbit. The LSI equations have translation and phase symmetries, i.e. if $(\phi(x,t), \psi(x,t), w(x,t))$ solves the LSI equations, then $(\mathrm{e}^{i\theta_1}\phi(x+x_0,t), \mathrm{e}^{i\theta_2}\psi(x+x_0,t), w(x+x_0,t))$ solves the same system for any $x_0 \in \mathbb{R}$ and $\theta_1, \theta_2 \in [0, 2\pi)$. We define the orbit $\mathrm{O}(f,g,h)$ of the triplet (f,g,h) as follows:

$$\mathrm{O}(f,g,h) = \{\mathrm{e}^{i\theta_1} f(.+x_0), \mathrm{e}^{i\theta_2} g(.+x_0), h(.+x_0); \theta_1, \theta_2 \in [0,2\pi), x_0 \in \mathbb{R}\}.$$

A solitary wave is said to be 'orbitally stable' if, for the initial data being near the solitary wave orbit, the solution at all later times remains near the solitary wave orbit.

The main result of this section is the following theorem.

THEOREM 2 For $c > 0$ and $4\omega - c^2 > 0$, solitary wave solution of (1.2)

$$\left.\begin{aligned} \mathrm{e}^{i\omega t}\Phi(x-ct) &= \mathrm{e}^{i\omega t} R_1(x-ct)\mathrm{e}^{i\frac{c(x-ct)}{2}}, \\ \mathrm{e}^{i\omega t}\Psi(x-ct) &= \mathrm{e}^{i\omega t} R_2(x-ct)\mathrm{e}^{i\frac{c(x-ct)}{2}}, \\ W(x-ct) &= -\tfrac{\beta}{c}[R_1^2(x-ct) + R_2^2(x-ct)], \end{aligned}\right\} \tag{4.1}$$

is orbitally stable, i.e. for any $\epsilon \geqslant 0$, there exists a corresponding $\delta \geqslant 0$ such that the initial data $(\phi_0, \psi_0, w_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with

$$\|\phi_0(.) - \Phi(.)\|_{H^1} \leqslant \delta, \quad \|\psi_0(.) - \Psi(.)\|_{H^1} \leqslant \delta, \quad \|w_0(.) - W(.)\|_2 \leqslant \delta,$$

imply

$$\inf_{\substack{x_0 \in \mathbb{R} \\ \theta_1 \in [0,2\pi)}} \|\mathrm{e}^{i\theta_1}\phi(.+x_0,t) - \Phi(.)\|_{H^1} \leqslant \epsilon,$$

$$\inf_{\substack{x_0\in\mathbb{R}\\ \theta_2\in[0,2\pi)}} \|e^{i\theta_2}\psi(.+x_0,t)-\Psi(.)\|_{H^1}\leqslant\epsilon,$$
$$\inf_{x_0\in\mathbb{R}}\|w(.+x_0,t)-W(.)\|_2\leqslant\epsilon.$$

In order to show that solitary waves (4.1) are orbitally stable, i.e. to prove Theorem 2; we have to find an estimate on the distance in $H^1(\mathbb{R})\times H^1(\mathbb{R})$ between the orbit $\mathrm{O}_{(R_1,R_2)}$ of solitary waves and the solution $(\phi(x,t),\psi(x,t))$ of the LSI system. The deviation of the solution $(\phi(x,t),\psi(x,t))$ corresponding to the initial data (ϕ_0,ψ_0) from the orbit $\mathrm{O}_{(R_1,R_2)}$ of solitary waves is measured by the metric

$$\rho_\Omega^2[(\phi,\psi),\mathrm{O}_{(R_1,R_2)}]=\inf_{\substack{x_0\in\mathbb{R}\\ \theta_1,\theta_2\in[0,2\pi)}}\{I_\Omega\},$$

where

$$\begin{aligned}I_\Omega(x_0,\theta_1,\theta_2)&=N_\Omega(e^{i\theta_1}e^{-i\frac{c}{2}(.+x_0-ct)}\phi(.+x_0,t)-R_1)\\&\quad+N_\Omega(e^{i\theta_2}e^{-i\frac{c}{2}(.+x_0-ct)}\psi(.+x_0,t)-R_2).\end{aligned}\tag{4.2}$$

The norm function N_Ω in (4.2) is defined as $N_\Omega(f)=\Omega\|f\|_2^2+\|\nabla f\|_2^2$ and satisfies $\min(1,\Omega)\,\|f\|_{H^1}^2\leqslant N_\Omega(f)\leqslant\max(1,\Omega)\|f\|_{H^1}^2$. Perturbations of solitary waves, denoted by $w_1(x,t)$, $w_2(x,t)$ and $\eta(x,t)$, are defined in the form

$$w_1(x,t)=e^{i\theta_1}e^{-i\frac{c}{2}(x+x_0-ct)}\phi(x+x_0,t)-R_1(x),\tag{4.3}$$
$$w_2(x,t)=e^{i\theta_2}e^{-i\frac{c}{2}(x+x_0-ct)}\psi(x+x_0,t)-R_2(x),\tag{4.4}$$
$$\eta(x,t)=\omega(x+x_0,t)+\frac{\beta}{c}[R_1^2(x)+R_2^2(x)],\tag{4.5}$$

where $w_k(x,t)=p_k(x,t)+iq_k(x,t)(k=1,2)$ are complex-valued functions and $\eta(x,t)$ is a real-valued function. Here, θ_1,θ_2 and x_0 will be chosen later where the infimum of I_Ω is attained. Equations (4.2) and (4.3–4.5) show that we have to find estimates on the H^1-norms of $w_1(x,t)$ and $w_2(x,t)$ and the L^2-norm of $\eta(x,t)$.

The following lemma is a generalization of the one that was proved in the context of the orbital stability of solitary waves by Bona (1975) for the Korteweg-de Vries equation and by Angulo & Montenegro (2001) for the LSI equations with an integral term. The following lemma states that there are $\theta_i=\theta_i(t)(i=1,2)$ and $x_0=x_0(t)$ such that infimum of $I_\Omega(x_0,\theta_1,\theta_2)$ exists where the local well posedness of the Cauchy problem for (1.2) is used.

LEMMA 3 Let (ϕ,ψ,u) be a solution of (1.2) corresponding to the initial data $(\phi_0,\psi_0,u_0)\in H^1(\mathbb{R})\times H^1(\mathbb{R})\times L^2(\mathbb{R})$ with the properties $\|\phi_0\|_2=\|R_1\|_2$ and $\|\psi_0\|_2=\|R_2\|_2$. Suppose that $I_\Omega(x_0,\theta_1,\theta_2)<\Omega(\|R_1\|_2^2+\|R_2\|_2^2)$ for some $t_0\in[0,T]$ and some $(x_0,\theta_1,\theta_2)\in\mathbb{R}\times[0,2\pi)\times[0,2\pi)$. Then $\inf\{I_\Omega|x_0\in\mathbb{R},\theta_1,\theta_2\in[0,2\pi)\}$ is assumed at least once.

Proof. It is clear that I_Ω is a continuous function of (x_0,θ_1,θ_2) on $\mathbb{R}\times[0,2\pi)\times[0,2\pi)$. Moreover, for any $(\theta_1,\theta_2)\in[0,2\pi)\times[0,2\pi)$, we have

$$\lim_{x_0\to\mp\infty}I_\Omega(x_0,\theta_1,\theta_2)=\|[e^{-i\frac{c}{2}(\cdot-ct)}\phi(\cdot,t)]'\|_2^2++\|[e^{-i\frac{c}{2}(\cdot-ct)}\psi(\cdot,t)]'\|_2^2$$

$$+\|R_1'(\cdot)\|_2^2+\|R_2'(\cdot)\|_2^2+2\Omega\|R_1(\cdot)\|_2^2+2\Omega\|R_2(\cdot)\|_2^2,$$
$$=\left(\frac{7\Omega}{3}+\frac{c^2}{4}\right)(\|R_1\|_2^2+\|R_2\|_2^2), \tag{4.6}$$

where (3.3) is used. The hypothesis $I_\Omega(x_0,\theta_1,\theta_2)<\Omega(\|R_1\|_2^2+\|R_2\|_2^2)$, the continuity of I_Ω and (4.6) imply the result. □

We now show that the infimum of I_Ω is attained at a finite value of x_0 for some $t_0\in[0,T]$. For this aim, it will suffice to show that $I_\Omega(x_0,\theta_1,\theta_2)<\Omega(\|R_1\|_2^2+\|R_2\|_2^2)$ holds in some interval. Using the inequality $\|a+b\|_2^2\leqslant 2\|a\|_2^2+2\|b\|_2^2$, one can obtain

$$I_\Omega(ct,-\omega t,-\omega t)\leqslant 2\|\phi'(\cdot)-\phi_s{}'(\cdot)\|_2^2+\left(\frac{c^2}{2}+\Omega\right)\|\phi(\cdot)-\phi_s(\cdot)\|_2^2$$
$$+2\|\psi'(\cdot)-\psi_s{}'(\cdot)\|_2^2+\left(\frac{c^2}{2}+\Omega\right)\|\psi(\cdot)-\psi_s(\cdot)\|_2^2,$$

where prime denotes differentiation with respect to spatial variable x. Solitary wave solutions (ϕ_s,ψ_s) given in (1.3) are globally defined. Thus, it follows from the continuous dependence theory that for a $T>0$, there exists a $\delta>0$ such that if

$$\|\phi_0(.)-\mathrm{e}^{i\frac{c}{2}\cdot}R_1(.)\|_{H^1}<\delta \quad\text{and}\quad \|\psi_0(.)-\mathrm{e}^{i\frac{c}{2}\cdot}R_2(.)\|_{H^1}<\delta,$$

then the solution $(\phi(x,t),\psi(x,t))$ corresponding to the initial data $(\phi_0(x),\psi_0(x))$ exists at least for $0\leqslant t\leqslant T$. This solution also satisfies

$$\|\phi(\cdot,t)-\phi_s(\cdot,t)\|_{H^1}<\epsilon \quad\text{and}\quad \|\psi(\cdot,t)-\psi_s(\cdot,t)\|_{H^1}<\epsilon.$$

Using this result, we get $I_\Omega(ct,-\omega t,-\omega t)\leqslant 4\epsilon^2(1+\omega)$. Choosing $\epsilon^2<\Omega(\|R_1\|_2^2+\|R_2\|_2^2)/[4(1+\omega)]$, shows that the hypothesis of Lemma 3 is satisfied at least for $(\tilde{x}_0,\tilde{\theta}_1,\tilde{\theta}_2)=(ct,-\omega t,-\omega t)$, from which we get an upper bound for I_Ω.

As a result of Lemma 3, the following compatibility conditions are obtained for the real-valued increment functions $p_i(x,t)$ and $q_i(x,t)(i=1,2)$

$$\int_{\mathbb{R}}(R_1^2+R_2^2)R_1q_1\,\mathrm{d}x=0, \tag{4.7}$$
$$\int_{\mathbb{R}}(R_1^2+R_2^2)R_2q_2\,\mathrm{d}x=0, \tag{4.8}$$
$$\int_{\mathbb{R}}(R_1^2+R_2^2)\left(R_1\frac{\partial p_1}{\partial x}+R_2\frac{\partial p_2}{\partial x}\right)\mathrm{d}x=0. \tag{4.9}$$

The relations (4.7–4.9) are obtained by differentiating I_Ω defined in (4.2) with respect to θ_1,θ_2 and x_0, using system (1.4) and then evaluating the resulting equations at values (x_0,θ_1,θ_2) which minimize I_Ω. Note that

$$I_\Omega=\|\mathrm{e}^{i\theta_1}A'-R_1'\|_2^2+\|\mathrm{e}^{i\theta_1}B'-R_2'\|_2^2+\Omega\|\mathrm{e}^{i\theta_1}A-R_1\|_2^2+\Omega\|\mathrm{e}^{i\theta_1}B-R_2\|_2^2,$$

where $\mathrm{e}^{i\theta_1}\mathrm{e}^{-i\frac{c}{2}(x+x_0-ct)}\phi(x+x_0,t) = \mathrm{e}^{i\theta_1}A(x+x_0,t) = R_1(x)+w_1(x,t)$ and $\mathrm{e}^{i\theta_2}\mathrm{e}^{-i\frac{c}{2}(x+x_0-ct)}\psi(x+x_0,t) = \mathrm{e}^{i\theta_2}B(x+x_0,t) = R_2(x)+w_2(x,t)$.

We now introduce a continuous non-linear functional L, called the Lyapunov functional, over $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R})$ in the form

$$L(\phi,\psi,\omega) = \omega(I_1+I_2) + \frac{c}{2}I_3 + I_4, \tag{4.10}$$

where $I_k (k = 1, 2, 3, 4)$, given in (2.2), are the conserved quantities of system (1.2). Thus, the Lyapunov functional is invariant with time, $\varDelta L(0) = \varDelta L(t)$. Our stability result will rely on the inequalities

$$\begin{aligned} &\varDelta L(0) \leqslant 2g(\varepsilon),\\ &\varDelta L(t) \geqslant g(\|w_1\|_{H^1}) + g(\|w_2\|_{H^1}), \end{aligned}$$

where $g(x) = a_1x^2 - a_2x^3 - a_3x^4$ for some positive constants $a_i (i = 1, 2, 3)$ and $\|w_i\|_{H^1} (i = 1, 2)$ is the distance between the solitary wave $(\varPhi, \varPsi)$ and the solution (ϕ, ψ) of (1.2). To find the bounds, we calculate $\varDelta L(t)$

$$\begin{aligned} \varDelta L(t) &= L(\phi(x,t), \psi(x,t), \omega(x,t)) - L(\varPhi(x), \varPsi(x), W(x)),\\ &= L(\varPhi(x) + \mathrm{e}^{i\frac{cx}{2}}w_1(x,t), \varPsi(x) + \mathrm{e}^{i\frac{cx}{2}}w_2(x,t), W(x) + \eta(x,t)) - L(\varPhi(x), \varPsi(x), W(x)). \end{aligned}$$

Expanding the functional L near $(\varPhi, \varPsi)$ yields

$$\varDelta L(t) = \delta L + \delta^2 L + \delta^3 L, \tag{4.11}$$

where δL, $\delta^2 L$ and $\delta^3 L$ are the first, second and third variations of L, respectively; and all variations higher than third order are zero. The explicit forms of variations are given as

$$\delta L = \int_{\mathbb{R}} 2\{[R_{1,xx} - \varOmega R_1 + \gamma(R_1^2+R_2^2)R_1]p_1 + [R_{2,xx} - \varOmega R_2 + \gamma(R_1^2+R_2^2)R_2]p_2\}\mathrm{d}x, \tag{4.12}$$

$$\begin{aligned} \delta^2 L = \int_{\mathbb{R}} \Big[&\frac{c}{2}\eta^2 + p_{1,x}^2 + q_{1,x}^2 + p_{2,x}^2 + q_{2,x}^2 + \varOmega(p_1^2+q_1^2+p_2^2+q_2^2)\\ &+2\beta(R_1p_1+R_2p_2)\eta - \gamma(R_1^2+R_2^2)(p_1^2+q_1^2+p_2^2+q_2^2)\Big]\,\mathrm{d}x, \end{aligned} \tag{4.13}$$

$$\delta^3 L = \int_{\mathbb{R}} \beta(p_1^2+q_1^2+p_2^2+q_2^2)\eta\,\mathrm{d}x, \tag{4.14}$$

where the relations $\varPhi(x) = R_1(x)\mathrm{e}^{\frac{icx}{2}}$, $\varPsi(x) = R_2(x)\mathrm{e}^{\frac{icx}{2}}$, $W(x) = -\beta^2(R_1^2(x) + R_2^2(x))/c$ and $w_k(x) = p_k(x) + iq_k(x) (k = 1, 2)$ are used. Because (R_1, R_2) is a solution (1.4), the first variation (4.12) vanishes. Thus, (R_1, R_2) is also a critical point of the Lyapunov functional L. From (4.13) and (4.14), we have

$$\begin{aligned} \varDelta L(t) = {} & \langle L_0q_1, q_1\rangle + \langle L_0q_2, q_2\rangle + \langle L_1p_1, p_1\rangle + \langle L_2p_2, p_2\rangle + 2\langle L_3p_1, p_2\rangle\\ & -\gamma\int_{\mathbb{R}}\left[\frac{1}{2}(p_1^2+q_1^2+p_2^2+q_2^2)^2 + 2(p_1^2+q_1^2+p_2^2+q_2^2)(p_1R_1+p_2R_2)\right]\mathrm{d}x\\ & +\frac{c}{2}\int_{\mathbb{R}}\left[\eta + \frac{2\beta}{c}(p_1R_1+p_2R_2) + \frac{\beta}{c}(p_1^2+q_1^2+p_2^2+q_2^2)\right]^2\mathrm{d}x, \end{aligned} \tag{4.15}$$

where the operators $L_i (i = 0, 1, 2, 3)$ are defined as

$$L_0 = -\frac{\partial^2}{\partial x^2} + \Omega - \gamma(R_1^2 + R_2^2), \qquad L_1 = -\frac{\partial^2}{\partial x^2} + \Omega - \gamma(3R_1^2 + R_2^2),$$
$$L_2 = -\frac{\partial^2}{\partial x^2} + \Omega - \gamma(R_1^2 + 3R_2^2), \quad L_3 = -2\gamma R_1 R_2.$$

We use the following lemmas to find a lower bound for $\Delta L(t)$.

LEMMA 4 There exist positive constants C_i $(i = 1, 2)$ such that

$$\langle L_0 q_i, q_i\rangle \geqslant C_i \|q_i\|_{H^1}^2 \quad (i = 1, 2). \tag{4.16}$$

Proof. It should be noted that $L_0 R_i = 0$ and $R_i > 0 (i = 1, 2)$. Therefore, L_0 is a non-negative operator, i.e. $\mu_i = \inf(\langle L_0 q_i, q_i\rangle / \langle q_i, q_i\rangle) \geqslant 0 (i = 1, 2)$. If the infimum of the functional μ_i subject to the constraints (4.7) and (4.8) is zero then it is attained at $q_i(x) = R_i(x)$. This contradicts to the above constraints, thus $\mu_i > 0 (i = 1, 2)$, i.e.

$$\langle L_0 q_i, q_i\rangle = \frac{1}{k_i + 1}\|q_i\| - \gamma \int_{\mathbb{R}} (R_1^2 + R_2^2) q_1^2 \, \mathrm{d}x + \frac{k_i}{k_i + 1}\|q_i\| \geqslant \bar{C}_i \|q_i\|_2^2 \quad (i = 1, 2),$$

where $\|q_i\| = \|\nabla q_i\|_2^2 + \Omega \|q_i\|_2^2$, k_i and $\bar{C}_i$ are some positive constants. If $k_i < \bar{C}_i / (2\gamma E^2)$, where $E = \max(\|R_1\|_\infty, \|R_2\|_\infty)$, then we have $\|q_i\|/(k_i + 1) - \gamma \int_{\mathbb{R}} (R_1^2 + R_2^2) q_1^2 \, \mathrm{d}x > 0$, and consequently

$$\langle L_0 q_i, q_i\rangle \geqslant C_i \|q_i\|_{H^1}^2 \quad (i = 1, 2),$$

where $C_i = k_i \min(1, \Omega)/(k_i + 1)$. □

To find a lower bound for the expression $\langle L_1 p_1, p_1\rangle + \langle L_2 p_2, p_2\rangle + 2\langle L_3 p_1, p_2\rangle$ in (4.15) is more difficult than that of $\langle L_0 q_i, q_i\rangle$. We will use the facts that (R_1, R_2) is the minimizer of the functional $J(u, v)$ and that the expression $\langle L_1 p_1, p_1\rangle + \langle L_2 p_2, p_2\rangle + 2\langle L_3 p_1, p_2\rangle$ is associated with the second variation of $J(u, v)$. First, we prove the following lemma which is a generalization of the one given in Weinstein (1985).

LEMMA 5 $\displaystyle\inf_{\substack{\langle f, R_1\rangle = 0 \\ \langle g, R_2\rangle = 0}} (\langle L_1 f, f\rangle + \langle L_2 g, g\rangle + 2\langle L_3 f, g\rangle) = 0.$

Proof. Recall that (R_1, R_2) is a minimizer of the non-linear functional $J(u, v)$. Thus, $\delta^2 J \geqslant 0$ near (R_1, R_2). The second variation of the functional J is of the form

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\epsilon^2} J(R_1 + \epsilon\eta_1,\ R_2 + \epsilon\eta_2)\,|_{\epsilon=0} &= a^2\,(\langle L_1\eta_1, \eta_1\rangle + \langle L_2\eta_2, \eta_2\rangle + 2\langle L_3\eta_1, \eta_2\rangle) \\
&\quad + a^2\left[\frac{\Omega^2}{3d}(\langle R_1,\ \eta_1\rangle + \langle R_2,\ \eta_2\rangle)^2\right. \\
&\quad \left. + \frac{2\Omega}{d}(\langle R_1,\ \eta_1\rangle + \langle R_2,\ \eta_2\rangle)(\langle R_{1,x},\ \eta_{1,x}\rangle + \langle R_{2,x},\ \eta_{2,x}\rangle)\right] \\
&\quad - \frac{a^2}{d}\left(\langle R_{1,x},\ \eta_{1,x}\rangle + \langle R_{2,x},\ \eta_{2,x}\rangle\right)^2 \geqslant 0,
\end{aligned} \tag{4.17}$$

where $a^2 = [27\gamma^2/(\Omega^3 d^6)]^{1/8}/(4\sqrt{2})$ and $d = \int_{\mathbb{R}}(u_x^2 + v_x^2)\mathrm{d}x$. It should be noted that (1.4) and Pohozaev-type identities given by (3.3) are used in obtaining (4.17).

If the increment functions are chosen as $\eta_1 = f$ and $\eta_2 = g$ with the properties $\langle f, R_1\rangle = 0$ and $\langle g, R_2\rangle = 0$, then it follows from (4.17) that

$$\langle L_1 f, f\rangle + \langle L_2 g, g\rangle + 2\langle L_3 f, g\rangle \geqslant 0. \tag{4.18}$$

Moreover, the functions $R_{1,x}$ and $R_{2,x}$ satisfy

$$\left.\begin{aligned} L_1 R_{1,x} + L_3 R_{2,x} &= (-R_{1,xx} + \Omega R_1 - \gamma(R_1^2 + R_2^2)R_1)_x = 0,\\ L_2 R_{2,x} + L_3 R_{1,x} &= (-R_{2,xx} + \Omega R_2 - \gamma(R_1^2 + R_2^2)R_2)_x = 0.\end{aligned}\right\} \tag{4.19}$$

As a result of (4.19), we find

$$\begin{aligned}\langle L_1 R_{1,x}, R_{1,x}\rangle &+ \langle L_2 R_{2,x}, R_{2,x}\rangle + \langle L_3 R_{1,x}, R_{2,x}\rangle + \langle L_3 R_{2,x}, R_{1,x}\rangle\\ &= \langle L_1 R_{1,x} + L_3 R_{2,x}, R_{1,x}\rangle + \langle L_2 R_{2,x} + L_3 R_{1,x}, R_{2,x}\rangle = 0,\end{aligned}$$

which shows that the infimum of (4.18) is assumed at $(R_{1,x}, R_{2,x})$. Because $f = R_{1,x}$ and $g = R_{2,x}$ satisfy the hypothesis of the lemma, we get $\langle f, R_1\rangle = \langle R_{1,x}, R_1\rangle = 0$ and $\langle g, R_2\rangle = \langle R_{2,x}, R_2\rangle = 0$. This completes the proof. □

In order to find a lower bound for $\langle L_1 p_1, p_1\rangle + \langle L_2 p_2, p_2\rangle + 2\langle L_3 p_1, p_2\rangle$, we require that the perturbed solution has the same L^2-norm as the solitary wave, as given in the hypotheses of Lemma 3,

$$\|\phi\|_2 = \|R_1\|_2, \quad \|\psi\|_2 = \|R_2\|_2. \tag{4.20}$$

Conditions (4.20) give rise to the following constraints

$$\langle R_i, p_i\rangle = -\frac{1}{2}[\langle p_i, p_i\rangle + \langle q_i, q_i\rangle] = -\frac{1}{2}\|w_i\|_2^2 < 0 \quad (i = 1, 2), \tag{4.21}$$

where definitions (4.3) are used. The restrictions (4.20) will be relaxed later and the stability of solitary waves will be proved with respect to general perturbations. To this end, we assume that the real parts of the increment functions, $p_i(x, t)$ $(i = 1, 2)$, will be of the form $p_i = p_{i\parallel} + p_{i\perp}$, where

$$p_{i\parallel} = \frac{\langle p_i, R_i\rangle}{\|R_i\|_2^2} R_i, \quad p_{i\perp} = p_i - \frac{\langle p_i, R_i\rangle}{\|R_i\|_2^2} R_i.$$

This gives rise to $\langle p_{i\perp}, R_i\rangle = 0$ $(i = 1, 2)$. Using the decomposition of $p_i(x, t)$ $(i = 1, 2)$, we have

$$\begin{aligned}&\langle L_1 p_1, p_1\rangle + \langle L_2 p_2, p_2\rangle + 2\langle L_3 p_1, p_2\rangle\\ &\quad = \langle L_1 p_{1\perp}, p_{1\perp}\rangle + \langle L_2 p_{2\perp}, p_{2\perp}\rangle + 2\langle L_3 p_{1\perp}, p_{2\perp}\rangle + \langle L_1 p_{1\parallel}, p_{1\parallel}\rangle + \langle L_2 p_{2\parallel}, p_{2\parallel}\rangle\\ &\qquad + 2\langle L_3 p_{1\parallel}, p_{2\parallel}\rangle + 2\langle L_1 p_{1\perp}, p_{1\parallel}\rangle + 2\langle L_2 p_{2\perp}, p_{2\parallel}\rangle + 2\langle L_3 p_{2\parallel}, p_{1\perp}\rangle + 2\langle L_3 p_{1\parallel}, p_{2\perp}\rangle.\end{aligned} \tag{4.22}$$

To find a suitable lower bound for $\langle L_1 p_{1\perp}, p_{1\perp}\rangle + \langle L_2 p_{2\perp}, p_{2\perp}\rangle + 2\langle L_3 p_{1\perp}, p_{2\perp}\rangle$ using Lemma 5, we further assume that $\langle p_1, R_1\rangle/\|R_1\|_2^2 = \langle p_2, R_2\rangle/\|R_2\|_2^2$. This condition appears as a result of non-linear coupling between ϕ and ψ.

LEMMA 6 There exist positive constants C_3 and C_4 such that

$$\begin{aligned}&\langle L_1 p_{1\perp}, p_{1\perp}\rangle + \langle L_2 p_{2\perp}, p_{2\perp}\rangle + 2\langle L_3 p_{1\perp}, p_{2\perp}\rangle\\ &\quad \geqslant C_3(\|p_1\|_2^2 + \|p_2\|_2^2) - C_4(\|w_1\|_{H^1}^4 - \|w_2\|_{H^1}^4).\end{aligned} \tag{4.23}$$

Proof. If $f = p_{1\perp}$ and $g = p_{2\perp}$ then the hypotheses of Lemma 5 are satisfied by $p_{1\perp}$ and $p_{2\perp}$. That is,

$$\langle L_1 p_{1\perp}, p_{1\perp}\rangle + \langle L_2 p_{2\perp}, p_{2\perp}\rangle + 2\langle L_3 p_{1\perp}, p_{2\perp}\rangle \geqslant 0. \tag{4.24}$$

If the infimum of (4.24) subject to the constraint (4.9) is zero then it is attained at $(p_{1\perp}, p_{2\perp}) = (R_{1,x}, R_{2,x})$. In such a case, for the increment functions $p_i = \alpha R_i + R_{i,x}$ $(i = 1, 2)$, where $\alpha = \langle p_i, R_i\rangle / \|R_i\|_2^2$ $(i = 1, 2)$, the constraint (4.9) reduces to

$$\frac{\alpha}{4}\int_{\mathbb{R}} [(R_1^2 + R_2^2)^2]_x \,\mathrm{d}x + \int_{\mathbb{R}} (R_1^2 + R_2^2)(R_1 R_{1xx} + R_2 R_{2xx})\mathrm{d}x = 0,$$

$$\int_{\mathbb{R}} \{[(R_1^2 + R_2^2)_x]^2 + 2(R_1^2 + R_2^2)[(R_{1,x})^2 + (R_{2,x})^2]\}\mathrm{d}x = 0,$$

where integration by parts is used. This result leads to $R_i = 0$ $(i = 1, 2)$ that contradicts positivity of ground state solutions (R_1, R_2). Thus, there exists a positive constant $\bar{C}_3$ such that

$$\langle L_1 p_{1\perp}, p_{1\perp}\rangle + \langle L_2 p_{2\perp}, p_{2\perp}\rangle + 2\langle L_3 p_{1\perp}, p_{2\perp}\rangle \geqslant \bar{C}_3. \tag{4.25}$$

Moreover, using $\langle p_{i\perp}, p_{i\perp}\rangle = \langle p_i, p_i\rangle - [\langle p_i, p_i\rangle + \langle q_i, q_i\rangle]^2/(4\|R_i\|_2^2)$, the inequality (4.25) can be arranged to yield (4.23)

$$\begin{aligned}\langle L_1 p_{1\perp}, p_{1\perp}\rangle + \langle L_2 p_{2\perp}, p_{2\perp}\rangle + 2\langle L_3 p_{1\perp}, p_{2\perp}\rangle &\geqslant C_3(\langle p_{1\perp}, p_{1\perp}\rangle + \langle p_{2\perp}, p_{2\perp}\rangle),\\ &= C_3\left(\|p_1\|_2^2 + \|p_2\|_2^2 - \frac{\|w_1\|_2^4}{4\|R_1\|_2^2} - \frac{\|w_2\|_2^4}{4\|R_2\|_2^2}\right),\\ &\geqslant C_3(\|p_1\|_2^2 + \|p_2\|_2^2) - C_4(\|w_1\|_{H^1}^4 + \|w_2\|_{H^1}^4),\end{aligned}$$

where continuous embedding of $H^1(\mathbb{R})$ in $L^4(\mathbb{R})$ is used and C_3 and C_4 are some positive constants. This completes the proof of Lemma 6. □

LEMMA 7 There exist positive constants C_5 and C_6 such that

$$\langle L_1 p_{1\|}, p_{1\|}\rangle + \langle L_2 p_{2\|}, p_{2\|}\rangle + 2\langle L_3 p_{1\|}, p_{2\|}\rangle \geqslant -C_5\|w_1\|_{H^1}^4 - C_6\|w_2\|_{H^1}^4. \tag{4.26}$$

Proof. Recall that $\langle L_i R_i, R_i\rangle = -2\gamma\langle R_i^2, R_i^2\rangle$ $(i = 1, 2)$. Firstly, using $p_{i\|} = \alpha R_i$ $(i = 1, 2)$, where $\alpha = -\|w_i\|_2^2/(2\|R_i\|_2^2)$, we obtain

$$\langle L_i p_{i\|}, p_{i\|}\rangle = \alpha^2\langle L_i R_i, R_i\rangle = -\frac{\gamma}{2}\frac{\|R_i^2\|_2^2}{\|R_i\|_2^4}\|w_i\|_2^4 \geqslant -\bar{C}_{4+i}\|w_i\|_{H^1}^4 \quad (i = 1, 2), \tag{4.27}$$

where $\bar{C}_5$ and $\bar{C}_6$ are positive constants. Secondly, using Sobolev embedding and Young's inequality $ab \leqslant a^p/p + b^q/q$ with $p = q = 2$, we obtain

$$\langle L_3 p_{1\|}, p_{2\|}\rangle = -\frac{\gamma}{4}\frac{\|R_1R_2\|_2^2}{\|R_1\|_2^2\|R_2\|_2^2}\|w_1\|_2^2\|w_2\|_2^2 \geqslant -\frac{\bar{C}_7}{2}(\|w_1\|_{H^1}^4 + \|w_2\|_{H^1}^4), \tag{4.28}$$

where $\bar{C}_7$ is a positive constant. (4.26) follows from (4.27) and (4.28). □

LEMMA 8 $\langle L_3 p_{1\|}, p_{2\perp}\rangle = 0$ and $\langle L_3 p_{2\|}, p_{1\perp}\rangle = 0$.

Proof. Using the definition of the operator L_3, we have $\langle L_3 p_{1\|}, p_{2\perp}\rangle = -2\gamma\alpha\langle R_1^2 p_{2\perp}, R_2\rangle$. Then

$$|\langle L_3 p_{1\|}, p_{2\perp}\rangle| \leqslant 2\gamma\alpha E^2|\langle p_{2\perp}, R_2\rangle| = 0 \tag{4.29}$$

and, similarly

$$|\langle L_3 p_{2\|}, p_{1\perp}\rangle| \leqslant 2\gamma\alpha E^2|\langle p_{1\perp}, R_1\rangle| = 0. \tag{4.30}$$

This completes the proof. □

LEMMA 9 There exist positive constants E_i and $F_i\,(i = 1, 2)$ such that

$$2\langle L_i p_{i\perp}, p_{i\|}\rangle \geqslant -E_i\|w_i\|_{H^1}^3 - F_i\|w_i\|_{H^1}^4 \quad (i = 1, 2). \tag{4.31}$$

Proof. For the terms $\langle L_i p_{i\perp}, p_{i\|}\rangle$, we find

$$\langle L_i p_{i\perp}, p_{i\|}\rangle = \alpha(\langle R_{i,x}, p_{i\perp,x}\rangle - 3\gamma\langle R_i^3, p_{i\perp}\rangle - \gamma\langle R_j^2 R_i, p_{i\perp}\rangle), \quad (i, j = 1, 2\ \ i \neq j), \tag{4.32}$$

where $\alpha = -\|w_i\|_2^2/(2\|R_i\|_2^2)$, $|\langle R_i^3, p_{i\perp}\rangle| \leqslant E^2|\langle R_i, p_{i\perp}\rangle| = 0$ and $|\langle R_j^2 R_i, p_{i\perp}\rangle| \leqslant E^2|\langle R_i, p_{i\perp}\rangle| = 0$. Using $p_{i\perp} = p_i - \alpha R_i$ and the Cauchy–Schwarz inequality in (4.32), we have

$$\begin{aligned}\langle L_i p_{i\perp}, p_{i\|}\rangle &\geqslant -\frac{\|w_i\|_2}{2\|R_i\|_2^2}\langle R_{i,x}, p_{i,x}\rangle - \frac{\|R_{i,x}\|_2^2}{4\|R_i\|_2^4}\|w_i\|_2^4,\\ &\geqslant -\frac{\|R_{i,x}\|_2}{2\|R_i\|_2^2}\|w_i\|_2^2\|w_{i,x}\|_2 - \frac{\|R_{i,x}\|_2^2}{4\|R_i\|_2^4}\|w_i\|_2^4 \quad (i = 1, 2).\end{aligned}$$

By continuous embedding of $H^1(\mathbb{R})$ in $L^2(\mathbb{R})$, the result follows

$$\langle L_i p_{i\perp}, p_{i\|}\rangle \geqslant -\frac{E_i}{2}\|w_i\|_{H^1}^3 - \frac{F_i}{2}\|w_i\|_{H^1}^4 \quad (i = 1, 2),$$

where E_i and F_i are some positive constants. □

LEMMA 10 There exist positive constants $A_i\ \ (i = 1, 2, 3)$ such that

$$\begin{aligned}\langle L_1 p_1, p_1\rangle + \langle L_2 p_2, p_2\rangle + 2\langle L_3 p_1, p_2\rangle \geqslant A_1(\|p_1\|_{H^1}^2 + \|p_2\|_{H^1}^2) - A_2(\|w_1\|_{H^1}^3 + \|w_2\|_{H^1}^3)\\ - A_3(\|w_1\|_{H^1}^4 + \|w_2\|_{H^1}^4).\end{aligned} \tag{4.33}$$

Proof. By direct computation, one can see that

$$\langle L_1 p_1, p_1\rangle + \langle L_2 p_2, p_2\rangle + 2\langle L_3 p_1, p_2\rangle = -\gamma \int_{\mathbb{R}} [(R_1^2 + R_2^2)(p_1^2 + p_2^2) + 2(R_1 P_1 + R_2 p_2)^2]\mathrm{d}x + \|p_1\| + \|p_2\|, \tag{4.34}$$

where $\|p_i\| = \Omega \|p_i\|_2^2 + \|\nabla p_i\|_2^2$ $(i = 1, 2)$. On the other hand, combining the inequalities (4.23), (4.26), (4.29), (4.30) and (4.31), we obtain

$$\langle L_1 p_1, p_1\rangle + \langle L_2 p_2, p_2\rangle + 2\langle L_3 p_1, p_2\rangle \geqslant C_3(\|p_1\|_2^2 + \|p_2\|_2^2) - E_1\|w_1\|_{H^1}^3 - E_2\|w_2\|_{H^1}^3 - C_8\|w_1\|_{H^1}^4 - C_9\|w_2\|_{H^1}^4, \tag{4.35}$$

where $C_8 = C_4 + C_5 + F_1$ and $C_9 = C_4 + C_6 + F_2$ are positive constants.

Using (4.34) and (4.35), for a sufficiently small positive number m, we find

$$\begin{aligned} I &= \frac{1}{m+1}(\|p_1\| + \|p_2\|) - \gamma \int_{\mathbb{R}} [(R_1^2 + R_2^2)(p_1^2 + p_2^2) + 2(R_1 P_1 + R_2 p_2)^2]\mathrm{d}x \\ &\geqslant \bar{C}_1(\|p_1\|_2^2 + \|p_2\|_2^2) - A_2(\|w_1\|_{H^1}^3 + \|w_2\|_{H^1}^3) - A_3(\|w_1\|_{H^1}^4 + \|w_2\|_{H^1}^4) \\ &\geqslant -A_2(\|w_1\|_{H^1}^3 + \|w_2\|_{H^1}^3) - A_3(\|w_1\|_{H^1}^4 + \|w_2\|_{H^1}^4), \end{aligned} \tag{4.36}$$

where $-\gamma \int_{\mathbb{R}} [(R_1^2 + R_2^2)(p_1^2 + p_2^2) + 2(R_1 P_1 + R_2 p_2)^2]\mathrm{d}x \geqslant -6\gamma E^2(\|p_1\|_2^2 + \|p_2\|_2^2)$ is used, and $\bar{C}_1 = (C_3 - 6\gamma m E^2)/(m+1)$, $A_2 = \max(E_1, E_2)/(m+1)$ and $A_3 = \max(C_8, C_9)/(m+1)$ are positive constants. Recalling that $\langle L_1 p_1, p_1\rangle + \langle L_2 p_2, p_2\rangle + 2\langle L_3 p_1, p_2\rangle = I + m(\|p_1\| + \|p_2\|)/(m+1)$, we obtain (4.33), where $A_1 = m\min(1, \Omega)/(1+m)$. This completes the proof of the lemma. □

Finally, the integral term in (4.15) can be estimated as

$$\begin{aligned} &\left| -\gamma \int_{\mathbb{R}} \left[\frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2)^2 + 2(p_1^2 + q_1^2 + p_2^2 + q_2^2)(p_1 R_1 + p_2 R_2)\right]\mathrm{d}x \right| \\ &\leqslant \bar{D}_1\|w_1\|_{H^1}(\|w_1\|_{H^1}^2 + \|w_2\|_{H^1}^2) + \bar{D}_2\|w_2\|_{H^1}(\|w_1\|_{H^1}^2 + \|w_2\|_{H^1}^2) + \gamma\|w_1\|_4^4 + \gamma\|w_2\|_4^4, \\ &\leqslant D_1\|w_1\|_{H^1}^3 + D_2\|w_2\|_{H^1}^3 + D_3\|w_1\|_{H^1}^4 + D_4\|w_2\|_{H^1}^4, \end{aligned} \tag{4.37}$$

where continuous embedding of $H^1(\mathbb{R})$ in $L^4(\mathbb{R})$ and in $L^\infty(\mathbb{R})$ and Young's inequality $ab \leqslant a^p/p + b^q/q$ with $p = 3$ and $q = 3/2$, are used, and $D_i (i = 1, 2, 3, 4)$ are positive constants.

Proof of Theorem 2. Combining the inequalities (4.16), (4.33) and (4.37), an upper bound for ΔL is given in terms of H^1-norms of the increment functions w_i as follows

$$\Delta L(t) \geqslant g(\|w_1\|_{H^1}) + g(\|w_2\|_{H^1}), \tag{4.38}$$

where $g(x) = a_1 x^2 - a_2 x^3 - a_3 x^4$ with positive constants

$$a_1 = \min(C_1, C_2, A_1), \quad a_2 = A_2 + \max(D_1, D_2), \quad a_3 = A_3 + \max(D_3, D_4).$$

Because $g(0) = 0$ and $g(x) \approx a_1 x^2$ near $x = 0$, there exists a positive number ϵ, $0 < \epsilon < \epsilon_0$, such that $g(x)$ increases on $[0, \epsilon_0]$. For such an ϵ, the inequalities

$$\|w_1(0)\|_{H^1} = \|\phi_0(.) - \Phi(.)\|_{H^1} \leqslant \delta, \quad \|w_2(0)\|_{H^1} = \|\psi_0(.) - \Psi(.)\|_{H^1} \leqslant \delta,$$

imply that

$$\varDelta L(0) < g(\epsilon) + g(\epsilon)$$

for sufficiently small δ. As $L(t)$ is invariant with time, i.e. $\varDelta L(t) = \varDelta L(0)$; from (4.38), we have

$$g(\|w_1(t)\|_{H^1}) + g(\|w_2(t)\|_{H^1}) \leqslant \varDelta L(t) = \varDelta L(0) < g(\epsilon) + g(\epsilon).$$

By continuity of the function g, there is at least a number $\epsilon \leqslant \epsilon_1 \leqslant \epsilon_0$ such that

$$\|w_1(t)\|_{H^1} \leqslant \epsilon_1 \leqslant c_1\epsilon \quad \text{and} \quad \|w_2(t)\|_{H^1} \leqslant \epsilon_1 \leqslant c_2\epsilon,$$

where $t \in [0, \infty)$ and $c_i (i = 1, 2)$ are positive constants.

Finally for the increment $\eta(x, t)$, we have to prove that $\|\eta(t)\|_2 \leqslant c\epsilon$ using the results obtained for $\|w_1(t)\|_{H^1}$ and $\|w_2(t)\|_{H^1}$. In (4.38), we have shown that

$$\begin{aligned}\varDelta L(t) &= K + \frac{c}{2}\int_{\mathbb{R}} \left[\eta + \frac{2\beta}{c}(p_1R_1 + p_2R_2) + \frac{\beta}{c}(p_1^2 + q_1^2 + p_2^2 + q_2^2)\right]^2 \mathrm{d}x \\ &\geqslant g(\|w_1(t)\|_{H^1}) + g(\|w_2(t)\|_{H^1}) \\ &\quad + \frac{c}{2}\int_{\mathbb{R}} \left[\eta + \frac{2\beta}{c}(p_1R_1 + p_2R_2) + \frac{\beta}{c}(p_1^2 + q_1^2 + p_2^2 + q_2^2)\right]^2 \mathrm{d}x,\end{aligned}$$

where

$$\begin{aligned}K = {} & \langle L_0q_1, q_1\rangle + \langle L_0q_2, q_2\rangle + \langle L_1p_1, p_1\rangle + \langle L_2p_2, p_2\rangle + 2\langle L_3p_1, p_2\rangle \\ & -\gamma\int_{\mathbb{R}} \left[\frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2)^2 + 2(p_1^2 + q_1^2 + p_2^2 + q_2^2)(p_1R_1 + p_2R_2)\right] \mathrm{d}x.\end{aligned}$$

For a given $\epsilon > 0$ with $0 < \epsilon < \epsilon_0$, the function g is increasing and $g(\|w_i(t)\|_{H^1}) > 0$ for $\|w_i(t)\|_{H^1} < c_i\epsilon (i = 1, 2)$. This shows that $K > 0$. By the invariance property of the functional L, $\varDelta L(t) = \varDelta L(0)$, we have

$$\int_{\mathbb{R}} \left[\eta + \frac{2\beta}{c}(p_1R_1 + p_2R_2) + \frac{\beta}{c}(p_1^2 + q_1^2 + p_2^2 + q_2^2)\right]^2 \mathrm{d}x \leqslant \frac{4}{c}g(\epsilon).$$

Using the inequalities $(a + b)^2 \geqslant \dfrac{a^2}{2} - b^2$ and $(a + b)^2 \leqslant 2(a^2 + b^2)$, we find

$$\|\eta(t)\|_2^2 \leqslant \frac{8}{c}g(\epsilon) + c_3(\|w_1(t)\|_{H^1}^2 + \|w_2(t)\|_{H^1}^2) + c_4(\|w_1(t)\|_{H^1}^4 + \|w_2(t)\|_{H^1}^2), \tag{4.39}$$

where the embedding of $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$ and $L^4(\mathbb{R})$ is used and c_3 and c_4 are positive constants. For some $c > 0$, we have $\|\eta(t)\|_2 \leqslant c\epsilon$. Thus, we have proved that solitary waves (ϕ_s, ψ_s, u_s) (1.3) are orbitally stable with respect to the small perturbations preserving the L^2-norms.

In order to prove stability of solitary waves with respect to general perturbations, we consider a solitary wave solution $(Q_{1\Omega}, Q_{2\Omega})$ which satisfy the system (1.4)

$$\begin{aligned}Q_{1\Omega}'' - \Omega Q_{1\Omega} + \gamma(Q_{1\Omega}^2 + Q_{2\Omega}^2)Q_{1\Omega} = 0,\\ Q_{2\Omega}'' - \Omega Q_{2\Omega} + \gamma(Q_{1\Omega}^2 + Q_{2\Omega}^2)Q_{2\Omega} = 0,\end{aligned}$$

where $\|\phi_0\|_2 \neq \|Q_{1\Omega}\|_2$ and $\|\psi_0\|_2 \neq \|Q_{2\Omega}\|_2$. Then, the functions $P_i(x) = Q_{i\Omega}(x/\sqrt{\Omega})/\sqrt{\Omega}$ $(i = 1, 2)$, satisfy

$$P_1'' - P_1 + \gamma(P_1^2 + P_2^2)P_1 = 0,$$
$$P_2'' - P_2 + \gamma(P_1^2 + P_2^2)P_2 = 0,$$

where $\|P_i\|_2 = \|Q_{i\Omega}\|_2/\sqrt[4]{\Omega}(i = 1, 2)$. Thus, for the solution $(Q_{1\Omega_0}, Q_{2\Omega_0})$ corresponding to $\Omega_0 > 0$, we have $\|P_i\|_2 = \|Q_{i\Omega_0}\|_2/\sqrt[4]{\Omega}_0$. It is possible to choose Ω_0 such that $\|\phi_0\|_2 = \|Q_{1\Omega_0}\|_2$ and $\|\psi_0\|_2 = \|Q_{2\Omega_0}\|_2$. In the proof of stability of solitary waves $(Q_{1\Omega}, Q_{2\Omega})$ relative to general perturbations that do not preserve L^2-norms, assuming the initial data obey the inequalities $\|\phi_0(.) - Q_{1\Omega}(.)e^{\frac{ic.}{2}}\|_{H^1} \leqslant \delta$ and $\|\psi_0(.) - Q_{2\Omega}(.)e^{\frac{ic.}{2}}\|_{H^1} \leqslant \delta$, the idea is to apply the preceding stability theory for $(Q_{1\Omega_0}, Q_{2\Omega_0})$ and then to use the triangle inequalities

$$\begin{aligned}\|e^{i\theta_1}\phi(.+x_0,t) - Q_{1\Omega}(.)e^{\frac{ic.}{2}}\|_{H^1} \leqslant{}& \|e^{i\theta_1}\phi(.+x_0,t) - Q_{1\Omega_0}(.)e^{\frac{ic.}{2}}\|_{H^1}\\ &+\|Q_{1\Omega_0}(.) - Q_{1\Omega}(.)\|_{H^1},\end{aligned} \tag{4.40}$$

$$\begin{aligned}\|e^{i\theta_2}\psi(.+x_0,t) - Q_{2\Omega}(.)e^{\frac{ic.}{2}}\|_{H^1} \leqslant{}& \|e^{i\theta_2}\psi(.+x_0,t) - Q_{2\Omega_0}(.)e^{\frac{ic.}{F2}}\|_{H^1}\\ &+\|Q_{2\Omega_0}(.) - Q_{2\Omega}(.)\|_{H^1}.\end{aligned} \tag{4.41}$$

The first terms in the right-hand side of the inequalities (4.40) and (4.41) are bounded from above by the orbital stability of the solutions $(Q_{1\Omega_0}, Q_{2\Omega_0})$. It remains to determine δ and to show that $\|Q_{i\Omega_0} - Q_{i\Omega}\|_{H^1}$ $(i = 1, 2)$ are also small. From the definitions of $Q_{i\Omega}$ and $Q_{i\Omega_0}$, we have

$$\begin{aligned}\|Q_{i\Omega} - Q_{i\Omega_0}\|_{H^1}^2 ={}& \sqrt{\Omega}\int_{\mathbb{R}}\left|P_i(x) - \sqrt{\frac{\Omega_0}{\Omega}}P_i\left(\sqrt{\frac{\Omega_0}{\Omega}}x\right)\right|^2 dx\\ &+\sqrt{\Omega^3}\int_{\mathbb{R}}\left|P_i'(x) - \frac{\Omega_0}{\Omega}P_i'\left(\sqrt{\frac{\Omega_0}{\Omega}}x\right)\right|^2 dx \quad (i = 1, 2).\end{aligned} \tag{4.42}$$

Using the inequality $(a - \epsilon b)^2 \leqslant 2\epsilon^2(a - b)^2 + 2(1 - \epsilon)^2a^2$, (4.42) is rewritten as

$$\begin{aligned}\|Q_{i\Omega} - Q_{i\Omega_0}\|_{H^1}^2 ={}& \sqrt{2}\Omega\left(\frac{\Omega_0}{\Omega}\int_{\mathbb{R}}\left|P_i(x) - P_i\left(\sqrt{\frac{\Omega_0}{\Omega}}x\right)\right|^2 dx + \left(\frac{\Omega_0}{\Omega} - 1\right)^2\int_{\mathbb{R}}P_i^2(x)\,dx\right)\\ &+2\sqrt{\Omega^3}\left(\frac{\Omega_0^2}{\Omega^2}\int_{\mathbb{R}}\left|P_i'(x) - P_i'\left(\sqrt{\frac{\Omega_0}{\Omega}}x\right)\right|^2 dx + \left(\frac{\Omega_0}{\Omega} - 1\right)^2\int_{\mathbb{R}}(P_i'(x))^2\,dx\right).\end{aligned} \tag{4.43}$$

Following the results of Angulo *et al.* (2002), obtained in the study of the stability of solitary waves in the critical case for a generalized Korteweg-de Vries equation and a generalized NLS equation, an upper bound for (4.43) can be given as follows

$$\|Q_{i\Omega_0} - Q_{i\Omega}\|_{H^1}^2 \leqslant G_i(\sqrt[4]{\Omega_0} - \sqrt[4]{\Omega})^2 + H_i(\sqrt{\Omega_0} - \sqrt{\Omega})^2 \quad (i = 1, 2),$$

where the fundamental theorem of calculus and Minkowski's inequality are used, and the positive constants G_i and $H_i (i = 1, 2)$ are given as

$$G_i = 8\sqrt{\frac{\Omega_0}{\Omega}}(\|xP_i'\|_2^2 + \Omega_0\|xP_i''\|_2^2) \quad H_i = \frac{2}{\sqrt{\Omega}}(\|P_i\|_2^2 + (\sqrt{\Omega}_0 + \sqrt{\Omega})^2\|P_i'\|_2^2).$$

We now show that there exists a positive constant $C = C(\Omega_0, P_i)$ such that $|\sqrt{\Omega_0} - \sqrt{\Omega}| \leqslant C\delta$ at least for small values of δ. Using the results

$$\sqrt{\Omega_0} = \frac{\|Q_{i\Omega_0}\|_2^2}{\|P_i\|_2^2} = \frac{\|\phi_0\|_2^2}{\|P_1\|_2^2} = \frac{\|\psi_0\|_2^2}{\|P_2\|_2^2}, \quad \sqrt{\Omega} = \frac{\|Q_{i\Omega}\|_2^2}{\|P_i\|_2^2},$$

we have

$$\begin{aligned} |\sqrt{\Omega_0} - \sqrt{\Omega}| &\leqslant \frac{1}{\|P_1\|_2^2}\left|\|\phi_0(.)\|_2^2 - \|Q_{1\Omega}(.)e^{\frac{ic.}{2}}\|_2^2\right| \\ &\leqslant \frac{1}{\|P_1\|_2^2}\left(\delta\|\phi_0(.)\|_2^2 + \left(1 + \frac{1}{\delta}\right)\|\phi_0(.) - Q_{1\Omega}(.)e^{\frac{ic.}{2}}\|_2^2\right), \end{aligned}$$

where the inequality $\left|\|a\|^2 - \|b\|^2\right| \leqslant \|a-b\|^2 + 2\|a\|\|a-b\|$ and Young's inequality are used. Using $\|\phi_0(.) - Q_{1\Omega}(.)e^{\frac{ic.}{2}}\|_2^2 \leqslant \delta^2$ and $\|\phi_0\|_2^2 = \sqrt{\Omega_0}\|P_1\|_2^2$, we have

$$|\sqrt{\Omega_0} - \sqrt{\Omega}| \leqslant \frac{1}{\|P_1\|_2^2}(\delta\sqrt{\Omega_0}\|P_1\|_2^2 + \delta^2 + \delta) \leqslant C(\Omega_0, P_1)\delta,$$

where $C(\Omega_0, P_i) = \sqrt{\Omega_0} + 2/\|P_1\|_2^2$. The inequality $|\sqrt{\Omega_0} - \sqrt{\Omega}| \leqslant C\delta$ implies $|\sqrt[4]{\Omega_0} - \sqrt[4]{\Omega}| \leqslant D\delta$ for some positive constant D. This completes the proof of Theorem 2.

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